FROM CALCULUS, BY MIKE SPIVAK

*CHAPTER 16 π IS IRRATIONAL

This short chapter, diverging from the main stream of the book, is included to demonstrate that we are already in a position to do some sophisticated mathematics. This entire chapter is devoted to an elementary proof that π is irrational. Like many "elementary" proofs of deep theorems, the motivation for many steps in our proof cannot be supplied; nevertheless, it is still quite possible to follow the proof step-by-step.

Two observations must be made before the proof. The first concerns the function

$$f_n(x) = \frac{x^n(1-x)^n}{n!},$$

which clearly satisfies

$$0 < f_n(x) < \frac{1}{n!} \quad \text{for } 0 < x < 1.$$

An important property of the function f_n is revealed by considering the expression obtained by actually multiplying out $x^n(1-x)^n$. The lowest power of x appearing will be n and the highest power will be $2n$. Thus f_n can be written in the form

$$f_n(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i,$$

where the numbers c_i are integers. It is clear from this expression that

$$f_n{}^{(k)}(0) = 0 \quad \text{if } k < n \text{ or } k > 2n.$$

Moreover,

$$f_n{}^{(n)}(x) = \frac{1}{n!}[n!\,c_n + \text{ terms involving } x]$$

$$f_n{}^{(n+1)}(x) = \frac{1}{n!}[(n+1)!\,c_{n+1} + \text{ terms involving } x]$$

$$\vdots$$

$$f_n{}^{(2n)}(x) = \frac{1}{n!}[(2n)!\,c_{2n}].$$

This means that

$$f_n^{(n)}(0) = c_n,$$
$$f_n^{(n+1)}(0) = (n+1)c_{n+1}$$
$$\vdots$$
$$f_n^{(2n)}(0) = (2n)(2n-1)\cdot\ldots\cdot(n+1)c_{2n},$$

where the numbers on the right are all integers. Thus

$f_n^{(k)}(0)$ is an *integer* for all k.

The relation

$$f_n(x) = f_n(1-x)$$

implies that

$$f_n^{(k)}(x) = (-1)^k f_n^{(k)}(1-x);$$

therefore,

$f_n^{(k)}(1)$ is also an *integer* for all k.

The proof that π is irrational requires one further observation: if a is any number, and $\varepsilon > 0$, then for sufficiently large n we will have

$$\frac{a^n}{n!} < \varepsilon.$$

To prove this, notice that if $n \geq 2a$, then

$$\frac{a^{n+1}}{(n+1)!} = \frac{a}{n+1}\cdot\frac{a^n}{n!} < \frac{1}{2}\cdot\frac{a^n}{n!}.$$

Now let n_0 be any natural number with $n_0 \geq 2a$. Then, whatever value

$$\frac{a^{n_0}}{(n_0)!}$$

may have, the succeeding values satisfy

$$\frac{a^{(n_0+1)}}{(n_0+1)!} < \frac{1}{2}\cdot\frac{a^{n_0}}{(n_0)!}$$
$$\frac{a^{(n_0+2)}}{(n_0+2)!} < \frac{1}{2}\cdot\frac{a^{(n_0+1)}}{(n_0+1)!} < \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{a^{n_0}}{(n_0)!}$$
$$\vdots$$
$$\frac{a^{(n_0+k)}}{(n_0+k)!} < \frac{1}{2^k}\cdot\frac{a^{n_0}}{(n_0)!}.$$

If k is so large that $\dfrac{a^{n_0}}{(n_0)!\,\varepsilon} < 2^k$, then

$$\frac{a^{(n_0+k)}}{(n_0+k)!} < \varepsilon,$$

which is the desired result. Having made these observations, we are ready for the one theorem in this chapter.

THEOREM 1 The number π is irrational; in fact, π^2 is irrational. (Notice that the irrationality of π^2 implies the irrationality of π, for if π were rational, then π^2 certainly would be.)

PROOF Suppose π^2 were rational, so that

$$\pi^2 = \frac{a}{b}$$

for some positive integers a and b. Let

(1) $$G(x) = b^n[\pi^{2n} f_n(x) - \pi^{2n-2} f_n{}''(x) + \pi^{2n-4} f_n{}^{(4)}(x) - \cdots + (-1)^n f_n{}^{(2n)}(x)].$$

Notice that each of the factors

$$b^n \pi^{2n-2k} = b^n (\pi^2)^{n-k} = b^n \left(\frac{a}{b}\right)^{n-k} = a^{n-k} b^k$$

is an integer. Since $f_n{}^{(k)}(0)$ and $f_n{}^{(k)}(1)$ are integers, this shows that

$$G(0) \text{ and } G(1) \text{ are integers.}$$

Differentiating G twice yields

(2) $$G''(x) = b^n[\pi^{2n} f_n{}''(x) - \pi^{2n-2} f_n{}^{(4)}(x) + \cdots + (-1)^n f_n{}^{(2n+2)}(x)].$$

The last term, $(-1)^n f_n{}^{(2n+2)}(x)$, is zero. Thus, adding (1) and (2) gives

(3) $$G''(x) + \pi^2 G(x) = b^n \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x).$$

Now let

$$H(x) = G'(x) \sin \pi x - \pi G(x) \cos \pi x.$$

Then

$$\begin{aligned} H'(x) &= \pi G'(x) \cos \pi x + G''(x) \sin \pi x - \pi G'(x) \cos \pi x + \pi^2 G(x) \sin \pi x \\ &= [G''(x) + \pi^2 G(x)] \sin \pi x \\ &= \pi^2 a^n f_n(x) \sin \pi x, \text{ by (3).} \end{aligned}$$

By the Second Fundamental Theorem of Calculus,

$$\begin{aligned} \pi^2 \int_0^1 a^n f_n(x) \sin \pi x \, dx &= H(1) - H(0) \\ &= G'(1) \sin \pi - \pi G(1) \cos \pi - G'(0) \sin 0 + \pi G(0) \cos 0 \\ &= \pi [G(1) + G(0)]. \end{aligned}$$

Thus

$$\pi \int_0^1 a^n f_n(x) \sin \pi x \, dx \quad \text{is an } \textit{integer}.$$

On the other hand, $0 < f_n(x) < 1/n!$ for $0 < x < 1$, so

$$0 < \pi a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!} \quad \text{for } 0 < x < 1.$$

Consequently,

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x \, dx < \frac{\pi a^n}{n!}.$$

This reasoning was completely independent of the value of n. Now if n is large enough, then

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x \, dx < \frac{\pi a^n}{n!} < 1.$$

But this is absurd, because the integral is an integer, and there is no integer between 0 and 1. Thus our original assumption must have been incorrect: π^2 is irrational. ∎

This proof is admittedly mysterious; perhaps most mysterious of all is the way that π enters the proof—it almost looks as if we have proved π irrational without ever mentioning a definition of π. A close reexamination of the proof will show that precisely one property of π is essential—

$$\sin(\pi) = 0.$$

The proof really depends on the properties of the function sin, and proves the irrationality of the smallest positive number x with $\sin x = 0$. In fact, very few properties of sin are required, namely,

$$\begin{aligned} \sin' &= \cos, \\ \cos' &= -\sin, \\ \sin(0) &= 0, \\ \cos(0) &= 1. \end{aligned}$$

Even this list could be shortened; as far as the proof is concerned, cos might just as well be defined as $\sin'$. The properties of sin required in the proof may then be written

$$\begin{aligned} \sin'' + \sin &= 0, \\ \sin(0) &= 0, \\ \sin'(0) &= 1. \end{aligned}$$

Of course, this is not really very surprising at all, since, as we have seen in the previous chapter, these properties characterize the function sin completely.