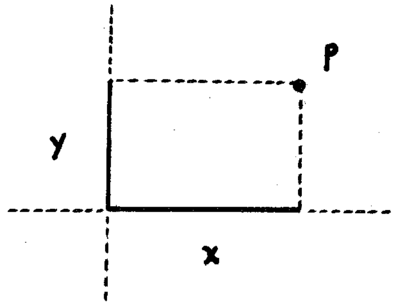
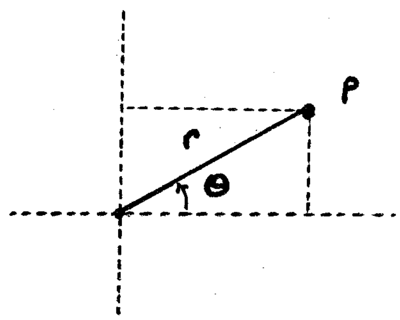
POLAR COORDINATES

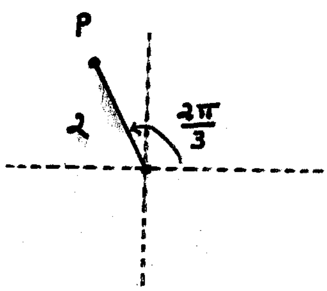
RECTANGULAR (CARTESIAN) COORDINATES LOCATE POINTS IN THE PLANE WITH A PAIR OF NUMBERS (x,y).



SOMETIMES IT IS CONVENIENT TO USE OTHER NUMBERS INSTEAD, E.G., POLAR COORDINATES (r,θ)



UNLIKE RECTANGULAR COORDINATES, THESE ARE NOT UNIQUE (A GIVEN P HAS LOTS OF POLAR COORDINATES), E.G.,



THIS P HAS POLAR COORDINATES

$$(r,\theta) = \left(2, \frac{2\pi}{3}\right)$$

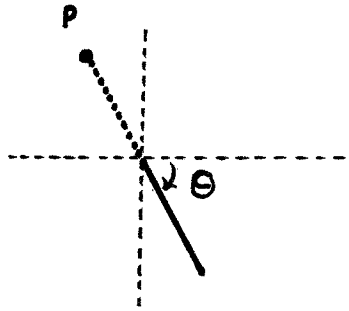
BUT ALSO

$$\left(2, \frac{2\pi}{3}+2\pi\right) = \left(2, \frac{8\pi}{3}\right)$$

$$\left(2, \frac{2\pi}{3}-4\pi\right) = \left(2, -\frac{10\pi}{3}\right)$$

ETC.

IT IS EVEN CONVENIENT TO ALLOW r TO BE NEGATIVE IN (r, θ) (MOVE BACKWARDS ALONG THE LINE THAT MAKES ANGLE θ WITH THE POSITIVE X-AXIS)



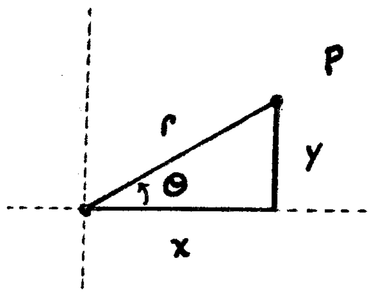
SO THIS SAME POINT ALSO HAS POLAR COORDINATES

$$(-2, -\tfrac{\pi}{3})$$

$$(-2, -\tfrac{\pi}{3} + 2\pi) = (-2, \tfrac{5\pi}{3})$$

ETC.

<u>NOTE</u> : THE POINT AT THE ORIGIN HAS $r = 0$, BUT θ IS TAKEN TO BE COMPLETELY ARBITRARY.



<u>TRANSFORMATION</u> <u>EQUATIONS</u> :

$\cos\theta = \frac{x}{r}$ AND $\sin\theta = \frac{y}{r}$ SO

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \quad \text{(PROVIDED } x \neq 0 \text{)}$$

EXAMPLES :

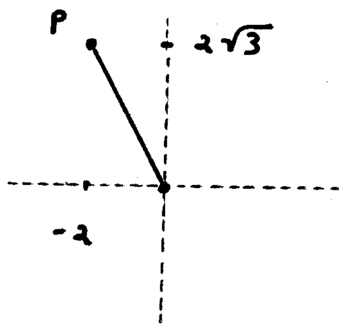
1. RECTANGULAR COORDINATES OF THE POINT P WITH POLAR COORDINATES $(r, \theta) = (6, \frac{2\pi}{3})$ ARE

$$x = r\cos\theta = 6\cos\frac{2\pi}{3} = 6(-\tfrac{1}{2}) = -3$$

$$y = r\sin\theta = 6\sin\frac{2\pi}{3} = 6(\tfrac{\sqrt{3}}{2}) = 3\sqrt{3}$$

2. ALL POSSIBLE POLAR COORDINATES OF THE POINT P WITH RECTANGULAR COORDINATES $(x, y) = (-2, 2\sqrt{3})$.

FOR THIS A PICTURE IS USEFUL (FOR FINDING θ).



$$r^2 = x^2 + y^2 = (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16$$

$$r = \pm 4$$

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

TO GET A VALUE OF θ NOTICE THAT P IS IN THE 2ND QUADRANT AND WRITE (TO JOG YOUR MEMORY)

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

THUS, THE θ BETWEEN 0 AND 2π THAT WORKS (HAS $\sin\theta = \frac{\sqrt{3}}{2}$ AND $\cos\theta = -\frac{1}{2}$) IS $\theta = \frac{2\pi}{3}$.

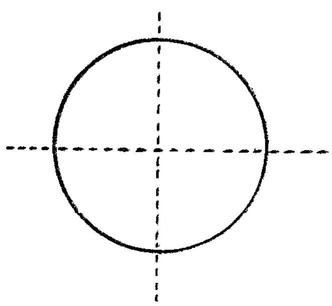
POLAR COORDINATES WITH $r = 4$:

$$(4, \tfrac{2\pi}{3})$$
$$(4, \tfrac{2\pi}{3} \pm 2\pi)$$
$$(4, \tfrac{2\pi}{3} \pm 4\pi)$$
$$\vdots$$
$$(4, \tfrac{2\pi}{3} + 2n\pi), \quad n = 0, \pm 1, \pm 2, \ldots$$

FOR POLAR COORDINATES WITH $r = -4$ WE MUST WALK BACKWARDS AND SO WE COULD TAKE $\theta = \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$, OR ANY OF THE FOLLOWING

$$(-4, \tfrac{5\pi}{3} + 2n\pi), \quad n = 0, \pm 1, \pm 2, \ldots$$

ONE OF THE MOST COMMON USES FOR POLAR COORDINATES IS WRITING EQUATIONS TO DESCRIBE CURVES THAT, IN RECTANGULAR COORDINATES, WOULD BE MORE COMPLICATED, E.G., THE UNIT CIRCLE



RECTANGULAR : $x^2 + y^2 = 1$ POLAR : $r = 1$

MOST OF THE EQUATIONS WHOSE GRAPHS COME UP IN PRACTICE ARE OF THE FORM

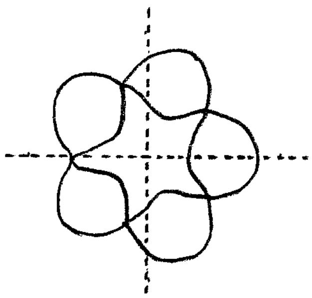
$$r = f(\theta)$$

THE <u>POLAR</u> <u>GRAPH</u> OF SUCH AN EQUATION IS THE SET OF ALL POINTS IN THE PLANE WITH <u>AT</u> <u>LEAST</u> <u>ONE</u> PAIR OF POLAR COORDINATES THAT SATISFY THE EQUATION, E.G., THE POINT WITH POLAR COORDINATES $(r, \theta) = (\frac{\pi}{4}, \frac{\pi}{4})$ SATISFIES THE EQUATION

$$r = \theta$$

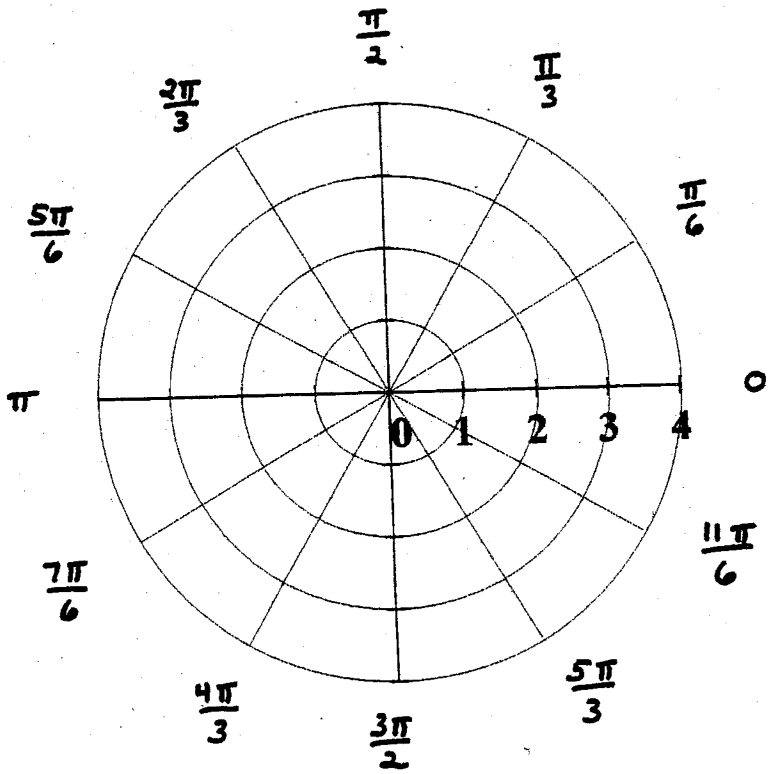
AND SO IS ON ITS GRAPH EVEN THOUGH THE SAME POINT HAS LOTS OF POLAR COORDINATES THAT DO NOT SATISFY THE EQUATION, E.G., $(r, \theta) = (\frac{\pi}{4}, \frac{\pi}{4} + 2\pi)$.

POLAR GRAPHS CAN BE EXTREMELY COMPLICATED, E.G., THE GRAPH OF $r = 2 + \cos\frac{5\theta}{2}$ IS

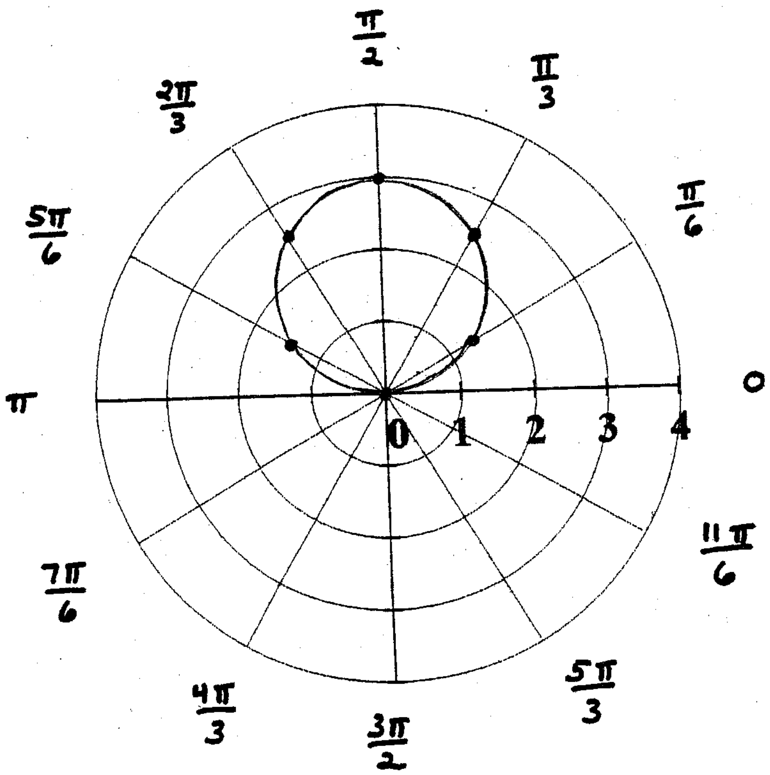


FORTUNATELY, THE GRAPHS WE NEED TO BE FAMILIAR WITH ARE OF JUST A FEW BASIC TYPES. WE'LL DO A FEW EXAMPLES AND, WHENEVER POSSIBLE, USE THE FOLLOWING TEMPLATE.

θ	r =
0	
π/6	
π/3	
π/2	
2π/3	
5π/6	
π	
7π/6	
4π/3	
3π/2	
5π/3	
11π/6	
2π	



θ	$r = 3\sin\theta$
0	$3(0) = 0$
$\pi/6$	$3(\frac{1}{2}) = \frac{3}{2}$
$\pi/3$	$3(\frac{\sqrt{3}}{2}) \approx 2.6$
$\pi/2$	$3(1) = 3$
$2\pi/3$	$3(\frac{\sqrt{3}}{2}) \approx 2.6$
$5\pi/6$	$3(\frac{1}{2}) = \frac{3}{2}$
π	$3(0) = 0$
$7\pi/6$	$3(-\frac{1}{2}) = -\frac{3}{2}$
$4\pi/3$	$3(-\frac{\sqrt{3}}{2}) \approx -2.6$
$3\pi/2$	$3(-1) = -3$
$5\pi/3$	$3(-\frac{\sqrt{3}}{2}) \approx -2.6$
$11\pi/6$	$3(-\frac{1}{2}) = -\frac{3}{2}$
2π	$3(0) = 0$



THIS SKETCH SUGGESTS THAT THE GRAPH IS A CIRCLE. IN THIS CASE WE CAN ACTUALLY VERIFY THAT THIS IS THE CASE IN THE FOLLOWING WAY :

$$r = 3 \sin \theta$$

$$r\,[\; r = 3 \sin \theta \;]$$

$$r^2 = 3 r \sin \theta$$

$$x^2 + y^2 = 3y$$

$$x^2 + y^2 - 3y = 0$$

$$x^2 + (y^2 - 3y \quad) = 0 \qquad \text{NOW COMPLETE THE SQUARE}$$

$$x^2 + (y^2 - 3y + \tfrac{9}{4}) = 0 + \tfrac{9}{4}$$

$$(x-0)^2 + (y - \tfrac{3}{2})^2 = (\tfrac{3}{2})^2$$

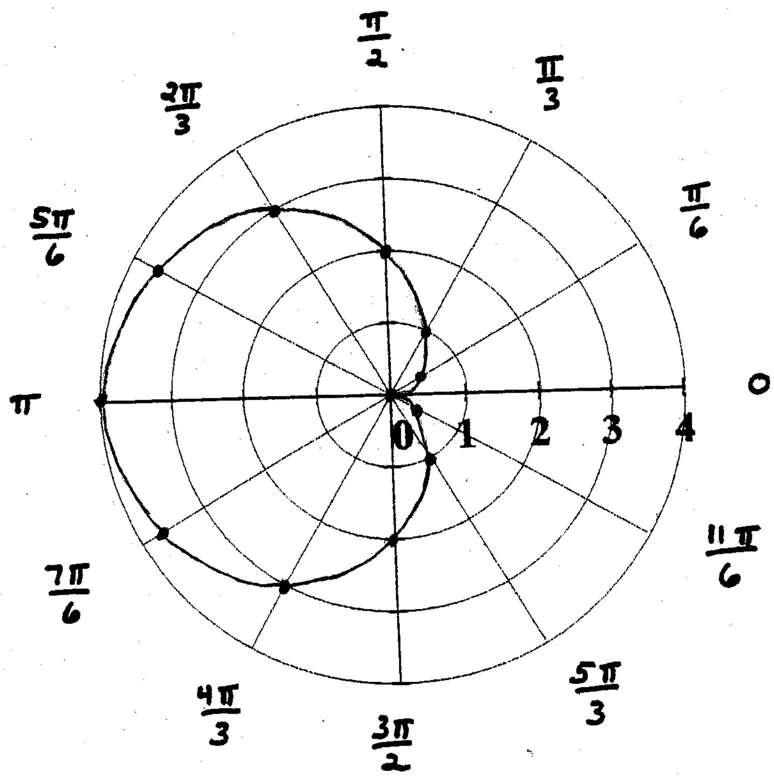
WHICH IS THE CIRCLE OF RADIUS $\frac{3}{2}$ ABOUT THE POINT $(0, \frac{3}{2})$.

IN GENERAL, IF a IS A POSITIVE CONSTANT, THE POLAR GRAPHS OF

$$r = a \quad , \quad r = a \sin \theta \quad , \quad r = a \cos \theta$$

ARE CIRCLES. TO FIND THE CENTER AND RADIUS IN THE LAST TWO CASES, COMPLETE THE SQUARE AS ABOVE.

θ	$r = 2(1-\cos\theta)$
0	$2(1-1) = 0$
$\pi/6$	$2(1-\frac{\sqrt{3}}{2}) \approx 0.28$
$\pi/3$	$2(1-\frac{1}{2}) = 1$
$\pi/2$	$2(1-0) = 2$
$2\pi/3$	$2(1+\frac{1}{2}) = 3$
$5\pi/6$	$2(1+\frac{\sqrt{3}}{2}) \approx 3.7$
π	$2(1+1) = 4$
$7\pi/6$	$2(1+\frac{\sqrt{3}}{2}) \approx 3.7$
$4\pi/3$	$2(1+\frac{1}{2}) = 3$
$3\pi/2$	$2(1-0) = 2$
$5\pi/3$	$2(1-\frac{1}{2}) = 1$
$11\pi/6$	$2(1-\frac{\sqrt{3}}{2}) \approx 0.28$
2π	$2(1-1) = 0$

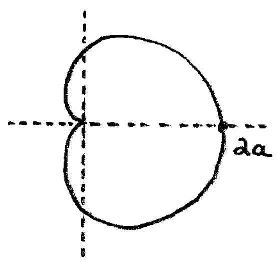


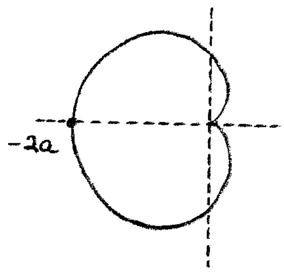
THE GRAPH IN THIS CASE IS CALLED A <u>CARDIOID</u>. IN GENERAL, THE POLAR GRAPHS OF

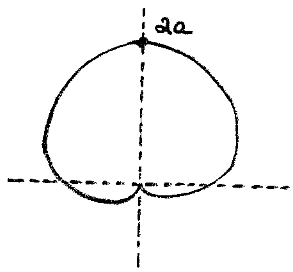
$$r = a(1+\cos\theta) \qquad r = a(1-\cos\theta)$$

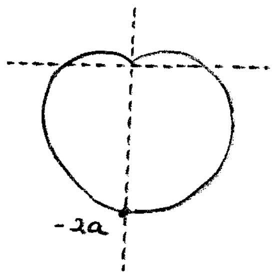
$$r = a(1+\sin\theta) \qquad r = a(1-\sin\theta)$$

ALL HAVE THIS GENERAL SHAPE, DIFFERING ONLY IN THEIR SIZE (DETERMINED BY a) AND ORIENTATION IN THE PLANE.









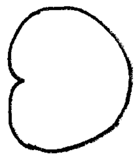
THE BEST PROCEDURE FOR GRAPHING ONE OF THESE IS TO REMEMBER WHAT SHAPE TO EXPECT, PLOT A FEW POINTS AND CONNECT THEM WITH A CARDIOID.

CARDIOIDS COME UP ALL THE TIME. A SIMILAR GROUP OF EQUATIONS THAT ONE SEES A BIT LESS FREQUENTLY IS

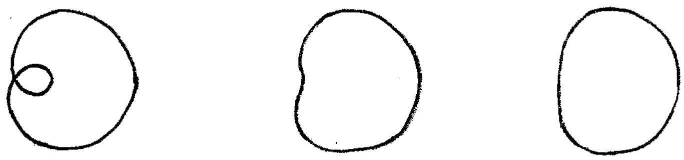
$$r = a + b\cos\theta \qquad r = a - b\cos\theta$$

$$r = a + b\sin\theta \qquad r = a - b\sin\theta$$

WHEN $b = a$ THESE ARE THE CARDIOIDS. WHEN $a \neq b$ THE GRAPH IS SIMILAR, BUT THE POINT



IS REPLACED BY A LOOP, A "DIMPLE", OR A FLAT SPOT.

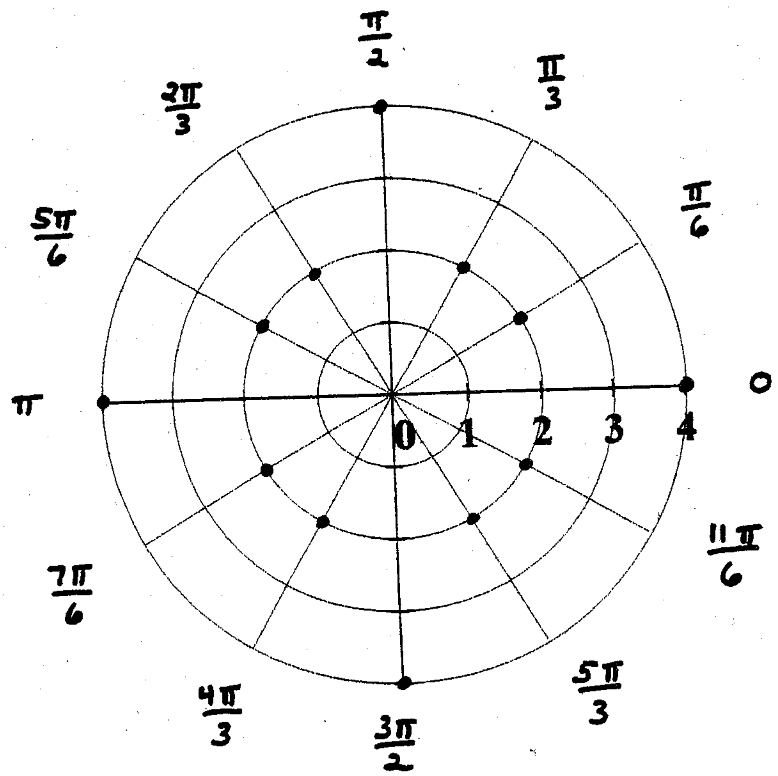


$\frac{a}{b} < 1$ $\qquad 1 < \frac{a}{b} < 2$ $\qquad \frac{a}{b} \geq 2$

AGAIN, THE BEST PROCEDURE IS JUST TO KNOW WHAT TO EXPECT AND THEN PLOT A FEW POINTS TO PIN DOWN THE DETAILS.

THE LAST THREE TYPES ARE CALLED LIMACONS.

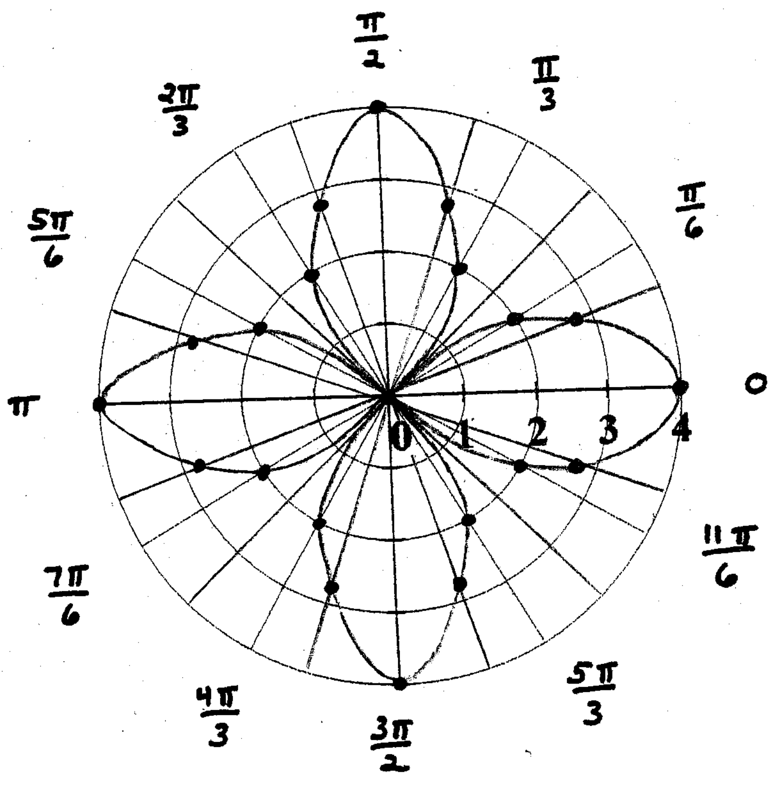
θ	$r = 4\cos 2\theta$	
0	$4\cos 0$	$= 4$
$\pi/6$	$4\cos \frac{\pi}{3}$	$= 2$
$\pi/3$	$4\cos \frac{2\pi}{3}$	$= -2$
$\pi/2$	$4\cos \pi$	$= -4$
$2\pi/3$	$4\cos \frac{4\pi}{3}$	$= -2$
$5\pi/6$	$4\cos \frac{5\pi}{3}$	$= 2$
π	$4\cos 2\pi$	$= 4$
$7\pi/6$	$4\cos \frac{7\pi}{3}$	$= 2$
$4\pi/3$	$4\cos \frac{8\pi}{3}$	$= -2$
$3\pi/2$	$4\cos 3\pi$	$= -4$
$5\pi/3$	$4\cos \frac{10\pi}{3}$	$= -2$
$11\pi/6$	$4\cos \frac{11\pi}{3}$	$= 2$
2π	$4\cos 4\pi$	$= 4$



CLEARLY NOT ENOUGH POINTS YET SO WE'LL COMPUTE SOME MORE.

NOTICE THAT $4\cos 2(-\theta) = 4\cos 2\theta$.

$\pm \pi/8$	$4\cos\frac{\pi}{4} = 2\sqrt{2} \approx 2.8$
$\pm \pi/4$	$4\cos\frac{\pi}{2} = 0$
$\pm 3\pi/8$	$4\cos\frac{3\pi}{4} = -2\sqrt{2} \approx -2.8$
$\pm 5\pi/8$	$4\cos\frac{5\pi}{4} = -2\sqrt{2} \approx -2.8$
$\pm 3\pi/4$	$4\cos\frac{3\pi}{2} = 0$
$\pm 7\pi/8$	$4\cos\frac{7\pi}{4} = 2\sqrt{2} \approx 2.8$



IN GENERAL, THE POLAR GRAPHS OF

$$r = a \cos n\theta \qquad\qquad r = a \sin n\theta$$

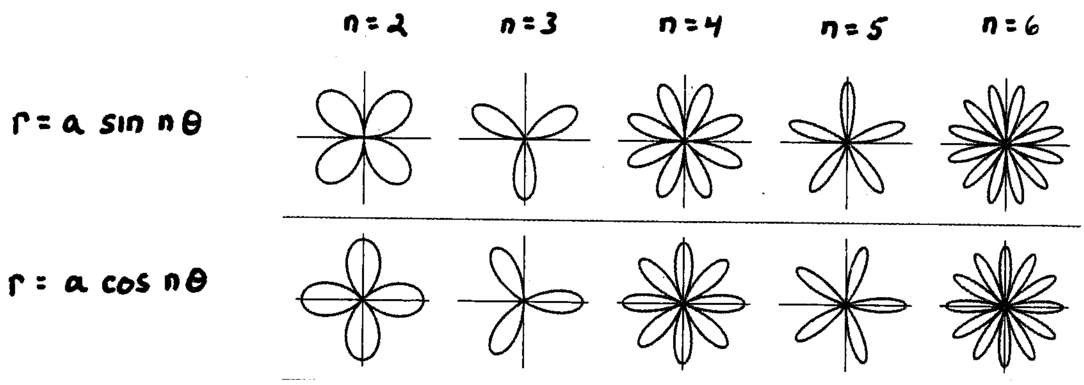
$$n \geq 2$$

ARE CALLED <u>ROSES</u> (I HAVE NO IDEA WHY SINCE THEY LOOK LIKE DAISIES) WITH

n <u>PETALS</u> IF n IS ODD

AND

$2n$ <u>PETALS</u> IF n IS EVEN



<u>PROCEDURE</u> : FIND THE VALUES OF θ FOR WHICH $\cos n\theta = \pm 1$ (OR $\sin n\theta = \pm 1$). THESE ARE THE "TIPS" OF THE PETALS.

FIND THE VALUES OF θ FOR WHICH $\cos n\theta = 0$ (OR $\sin n\theta = 0$).

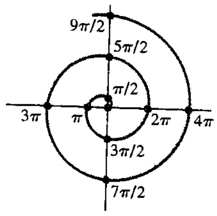
CONNECT WITH PETALS.

A FEW MISC. EXAMPLES / TECHNIQUES :

1. $r = \theta$, FOR $\theta \geq 0$

θ	r
0	0
$\frac{\pi}{2}$	$\frac{\pi}{2}$
π	π
$\frac{3\pi}{2}$	$\frac{3\pi}{2}$
2π	2π

ETC.



IN GENERAL, THE GRAPH OF $r = a\theta$ IS AN ARCHIMEDEAN SPIRAL.

SOMETIMES THE BEST WAY TO IDENTIFY A POLAR GRAPH IS TO CONVERT TO RECTANGULAR COORDINATES.

2. $r = 4 \sec\theta$: $r = 4\left(\frac{1}{\cos\theta}\right)$

$$r\cos\theta = 4$$
$$x = 4 \quad \text{(VERTICAL LINE)}$$

3. $r = 4\cos\theta + 4\sin\theta$

$$r^2 = 4r\cos\theta + 4r\sin\theta$$
$$x^2 + y^2 = 4x + 4y$$
$$(x^2 - 4x \quad) + (y^2 - 4y \quad) = 0$$
$$(x^2 - 4x + 4) + (y^2 - 4y + 4) = 4 + 4$$
$$(x-2)^2 + (y-2)^2 = 8$$

(CIRCLE OF RADIUS $2\sqrt{2}$ ABOUT $(2,2)$)