

POLYNOMIAL APPROXIMATION OF FUNCTIONS

SOME FUNCTIONS ARE SIMPLE, E.G.,

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

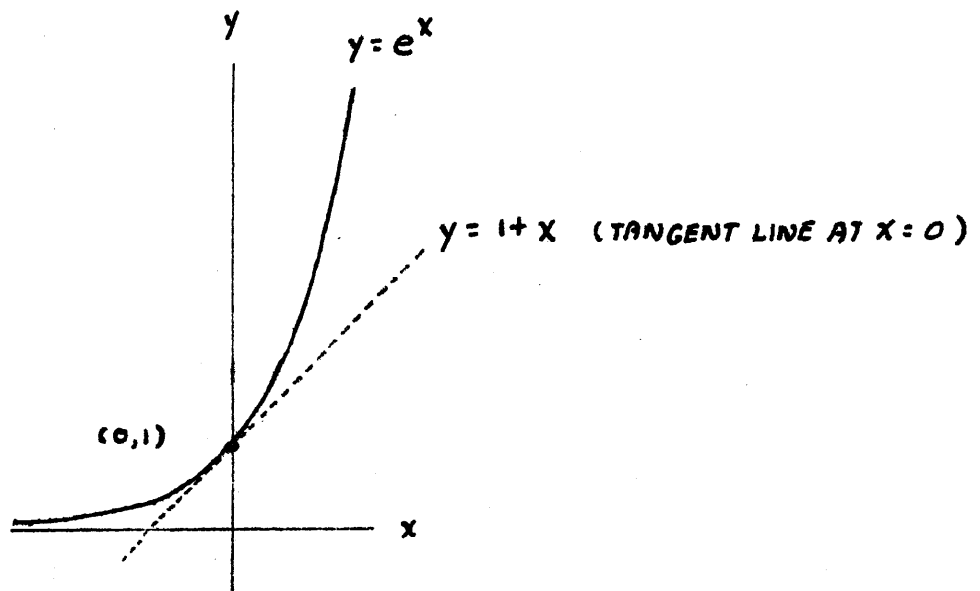
AND SOME ARE NOT, E.G.,

$$e^x$$

(FOR EXAMPLE, WHAT'S THE VALUE OF e^x AT $x = 0.1$?)

$$\begin{aligned} e^{0.1} &= e^{\frac{1}{10}} = \sqrt[10]{e} = \sqrt[10]{2.71828\dots} \\ &= ? \end{aligned}$$

IT WOULD BE NICE IF COMPLICATED FUNCTIONS COULD BE WELL APPROXIMATED BY SIMPLE FUNCTIONS, E.G.,



AT LEAST NEAR $x = 0$,

$$e^x \approx 1 + x.$$

WITH A LINEAR FUNCTION THIS IS THE BEST YOU CAN DO (TANGENT LINE GOES THROUGH THE SAME POINT $(0, 1)$ WITH THE SAME SLOPE 1 AS e^x)

WITH A QUADRATIC FUNCTION YOU CAN DO BETTER (SAME POINT, SAME SLOPE, SAME CONCAVITY)

$$y = a_0 + a_1 x + a_2 x^2$$

$$y(0) = a_0$$

$$y'(0) = a_1$$

$$y''(0) = 2a_2$$

CHOOSE THE COEFFICIENTS SO THAT THESE AGREE WITH e^x , $(e^x)'$ AND $(e^x)''$ AT $x = 0$, I.E.,

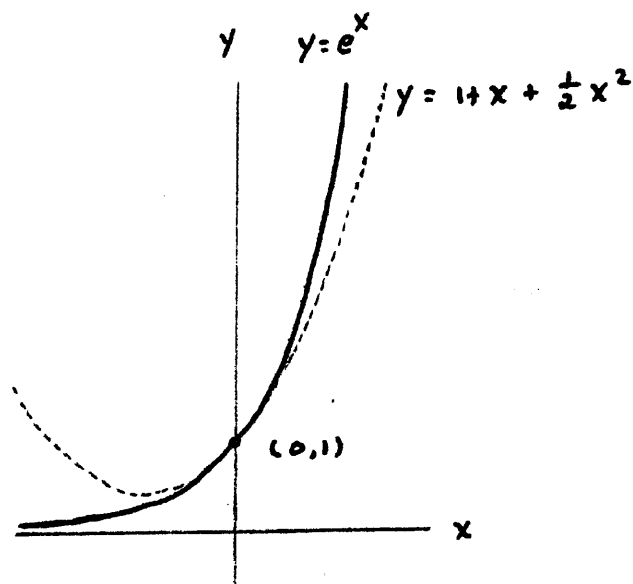
$$a_0 = 1$$

$$a_1 = 1$$

$$2a_2 = 1 \quad (\text{SO } a_2 = \frac{1}{2})$$

$$y = 1 + x + \frac{1}{2}x^2$$

IS THE LOCAL QUADRATIC APPROXIMATION TO e^x AT $x = 0$



WITH HIGHER DEGREE POLYNOMIALS WE CAN FORCE MORE DERIVATIVES AT $x = 0$ TO AGREE WITH THE DERIVATIVES OF e^x AT $x = 0$ AND SO GET A CLOSER FIT.

NOTE : $e^{0.1} \approx 1 + (0.1) + \frac{1}{2}(0.1)^2 = 1 + \frac{1}{10} + \frac{1}{200}$
 $= \frac{221}{200} = 1.105$

CALCULATOR : $e^{0.1} \approx 1.105170918$

LET'S DO THIS NOW FOR AN ARBITRARY FUNCTION $f(x)$: THE BEST n^{TH} DEGREE POLYNOMIAL APPROXIMATION TO $f(x)$ NEAR $x = 0$.

$$P_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$

$$P_n(0) = a_0$$

CHOOSE $a_0 = f(0)$

$$P_n'(0) = a_1$$

CHOOSE $a_1 = f'(0)$

$$P_n''(0) = 2a_2$$

CHOOSE $a_2 = \frac{f''(0)}{2}$

$$P_n^{(3)}(0) = 3 \cdot 2a_3$$

CHOOSE $a_3 = \frac{f^{(3)}(0)}{3 \cdot 2}$

$$a_4 = \frac{f^{(4)}(0)}{4 \cdot 3 \cdot 2}$$

⋮

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$\begin{aligned} P_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n \\ &= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k = n^{\text{TH}} \text{ MACLAURIN POLYNOMIAL OF } f(x) \end{aligned}$$

EXAMPLES:

1. FOR $f(x) = e^x$ ALL DERIVATIVES ARE ALSO e^x SO ALL OF THEIR VALUES AT $x=0$ ARE 1. THUS,

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n = \sum_{k=0}^n \frac{1}{k!}x^k$$

2. COMPUTE $P_7(x)$ FOR $f(x) = \sin x$.

$f(x) = \sin x$	$f(0) = 0$	$a_0 = 0$
$f'(x) = \cos x$	$f'(0) = 1$	$a_1 = 1$
$f''(x) = -\sin x$	$f''(0) = 0$	$a_2 = 0$
$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -1$	$a_3 = -\frac{1}{3!}$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$	$a_4 = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$	$a_5 = \frac{1}{5!}$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$	$a_6 = 0$
$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -1$	$a_7 = -\frac{1}{7!}$

$$\begin{aligned}
 P_7(x) &= 0 + 1x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 \\
 &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7
 \end{aligned}$$

NOTE: $\sin 1 \approx P_7(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040}$

$$= 0.841468254$$

CALCULATOR: $\sin 1 \approx 0.841470985$

CAN DO SUCH APPROXIMATIONS AT POINTS OTHER THAN $x=0$ ALSO.

BEST n^{TH} DEGREE POLYNOMIAL APPROXIMATION TO $f(x)$ NEAR $x=x_0$

IS THE TAYLOR POLYNOMIAL OF DEGREE n FOR $f(x)$ AT $x=x_0$

AND IS DEFINED AS FOLLOWS:

$$\begin{aligned}
 P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \\
 &\quad \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\
 &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad (x_0 = 0 \text{ GIVES MACLAURIN})
 \end{aligned}$$

EXAMPLE : FIND THE TAYLOR POLYNOMIAL OF DEGREE 5 FOR $f(x) = \cos x$
AT $x = \frac{\pi}{2}$.

$$\begin{array}{ll}
 f(x) = \cos x & f\left(\frac{\pi}{2}\right) = 0 \\
 f'(x) = -\sin x & f'\left(\frac{\pi}{2}\right) = -1 \\
 f''(x) = -\cos x & f''\left(\frac{\pi}{2}\right) = 0 \\
 f^{(3)}(x) = \sin x & f^{(3)}\left(\frac{\pi}{2}\right) = 1 \\
 f^{(4)}(x) = \cos x & f^{(4)}\left(\frac{\pi}{2}\right) = 0 \\
 f^{(5)}(x) = -\sin x & f^{(5)}\left(\frac{\pi}{2}\right) = -1
 \end{array}$$

$$\begin{aligned}
 P_5(x) &= 0 + (-1)(x - \frac{\pi}{2}) + 0 + \frac{1}{3!}(x - \frac{\pi}{2})^3 + 0 + \frac{-1}{5!}(x - \frac{\pi}{2})^5 \\
 &= -(x - \frac{\pi}{2}) + \frac{1}{3!}(x - \frac{\pi}{2})^3 - \frac{1}{5!}(x - \frac{\pi}{2})^5
 \end{aligned}$$

IT'S POSSIBLE TO MEASURE THE ACCURACY OF THESE POLYNOMIAL
APPROXIMATIONS :

GIVEN $f(x)$, x_0 AND n , LET

$$R_n(x) = f(x) - P_n(x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

= REMAINDER (ERROR) MADE IN APPROXIMATING
 $f(x)$ BY $P_n(x)$.

THEOREM: IF, ON SOME INTERVAL I CONTAINING x_0 ,

$$|f^{(n+1)}(x)| \leq M$$

FOR SOME CONSTANT M , THEN

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-x_0|^{n+1}$$

FOR ALL x IN I .

EXAMPLES:

1. EARLIER WE USED THE 7TH DEGREE MACLAURIN POLYNOMIAL TO APPROXIMATE $\sin 1$.

$$f(x) = \sin x$$

$$x_0 = 0$$

$$n = 7$$

$$x = 1$$

ON THE INTERVAL $I = [0, 1]$ (OR ANY OTHER INTERVAL)

$$|f^{(8)}(x)| \leq 1$$

THUS,

$$|R_7(1)| \leq \frac{1}{8!} (1-0)^8 = \frac{1}{8!} = \frac{1}{40320} = 0.000024802$$

2. EARLIER WE USED THE 2ND DEGREE TAYLOR POLYNOMIAL $P_2(x)$ FOR $f(x) = e^x$ TO APPROXIMATE $e^{0.1}$.

HOW LARGE MUST n BE CHOSEN IN ORDER FOR $P_n(0.1)$ TO APPROXIMATE $e^{0.1}$ TO FIVE DECIMAL PLACE ACCURACY, I.E., SO THAT

$$|R_n(0.1)| < 0.000005$$

$f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x$ AND SO, ON THE INTERVAL $I = [0, 0.1]$,

$$|f^{(n+1)}(x)| < 2$$

[$f^{(n+1)}(x) = e^x$ IS INCREASING ON $[0, 0.1]$ SO ITS LARGEST VALUE IS $e^{0.1}$ WHICH IS SURELY LESS THAN 2 BECAUSE $2^{10} > e$] THUS,

$$|R_n(0.1)| < \frac{2}{(n+1)!} (0.1-0)^{n+1} = \frac{2}{(n+1)!} \left(\frac{1}{10}\right)^{n+1}$$

THUS, WE WANT TO CHOOSE n SO THAT

$$\frac{2}{10^{n+1}(n+1)!} < 0.000005$$

$$\frac{1}{10^{n+1}(n+1)!} < 0.0000025$$

$$10^{n+1}(n+1)! > \frac{1}{0.0000025}$$

$$10^{n+1}(n+1)! > 400,000$$

CHECK A FEW VALUES OF n :

$$n = 2 : 10^3(3!) = 6000$$

$$n = 3 : 10^4(4!) = 240,000$$

$$n = 4 : 10^5(5!) = 12,000,000$$

SO $n = 4$ WILL DO.

IT WOULD BE PARTICULARLY NICE IF $|R_n(x)| \rightarrow 0$ AS $n \rightarrow \infty$

FOR THEN WE COULD GET ARBITRARILY GOOD APPROXIMATIONS JUST BY CHOOSING THE DEGREE HIGH ENOUGH.

WE RETURN TO THIS IN THE NEXT SECTION.