PRODUCTS AND TOPOLOGICAL GROUPS

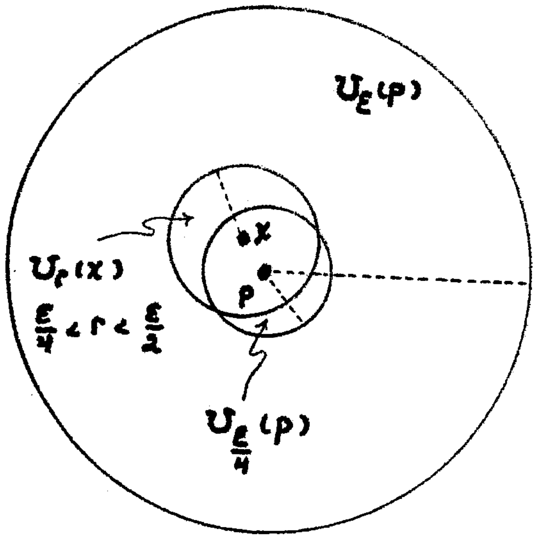
IN $\mathbb{R}^n$ EVERY OPEN SET CAN BE WRITTEN AS A UNION OF OPEN BALLS $U_\varepsilon(p)$ SO WE SAY THAT $\{U_\varepsilon(p) : p \in \mathbb{R}^n, \varepsilon > 0\}$ IS A "BASIS FOR THE OPEN SETS IN $\mathbb{R}^n$".

MORE GENERALLY : X A TOPOLOGICAL SPACE WITH TOPOLOGY $\mathcal{T}$. A SUBCOLLECTION $\mathcal{B}$ OF $\mathcal{T}$ IS A BASIS FOR $\mathcal{T}$ IF EVERY ELEMENT OF $\mathcal{T}$ CAN BE WRITTEN AS A UNION OF ELEMENTS OF $\mathcal{B}$.

THE TOPOLOGY OF $\mathbb{R}^n$ (AND EVERY SUBSPACE OF $\mathbb{R}^n$) HAS A COUNTABLE BASIS. FOR $\mathbb{R}^n$,

$$\{U_r(x) : r > 0,\ r \in \mathbb{Q},\ x = (x^1, \dots, x^n),\ x^i \in \mathbb{Q},\ i = 1, \dots, n\}$$

IS ONE SUCH BASIS :



$x = (x^1, \dots, x^n)$

$x^i \in \mathbb{Q}$

$p \in U_r(x) \subseteq U_\varepsilon(p)$

BY INTERSECTING THE ELEMENTS OF THIS BASIS WITH ANY NONEMPTY SUBSET X OF $\mathbb{R}^n$ WE OBTAIN A COUNTABLE BASIS FOR THE SUBSPACE TOPOLOGY OF X.

A TOPOLOGICAL SPACE IS <u>SECOND COUNTABLE</u> IF THERE IS A COUNTABLE BASIS FOR ITS TOPOLOGY.

A <u>TOPOLOGICAL MANIFOLD</u> IS A TOPOLOGICAL SPACE THAT IS HAUSDORFF, LOCALLY EUCLIDEAN AND SECOND COUNTABLE.

<u>EXAMPLES</u>: $\mathbb{R}^n$, S^n, $GL(n,\mathbb{R})$, $\mathbb{RP}^n$ (SECOND COUNTABILITY OF $\mathbb{RP}^n$ IS YOUR NEXT EXERCISE).

<u>EXERCISE</u> 29: SHOW THAT $\mathbb{RP}^n$ IS SECOND COUNTABLE.
<u>HINT</u>: SHOW, AS WE DID FOR $\mathbb{RP}^2$, THAT THERE ARE $n+1$ CHARTS $(U_1,\varphi_1),\dots,(U_{n+1},\varphi_{n+1})$ FOR $\mathbb{RP}^n$ WITH $\bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$ AND EACH U_i HOMEOMORPHIC TO AN OPEN SET IN $\mathbb{R}^n$.

NOW WE REVERSE OUR POINT OF VIEW. GIVEN A <u>SET</u> X AND A COLLECTION $\mathcal{B}$ OF SUBSETS OF X, WE ASK

> WHAT MUST BE TRUE OF $\mathcal{B}$ IN ORDER THAT IT BE A BASIS FOR <u>SOME</u> TOPOLOGY ON X?

<u>THEOREM</u>: LET X BE A SET AND $\mathcal{B}$ A COLLECTION OF SUBSETS OF X THAT SATISFIES

$$V, W \in \mathcal{B} \text{ AND } x \in V \cap W \Rightarrow \exists\, U \in \mathcal{B} \text{ S.T. }$$
$$x \in U \subseteq V \cap W.$$

THEN THE COLLECTION $\mathcal{T}_{\mathcal{B}}$ CONSISTING OF ϕ, X AND ALL UNIONS OF MEMBERS OF $\mathcal{B}$ IS A TOPOLOGY FOR X.

<u>PROOF</u>: MUST SHOW THAT $\phi, X \in \mathcal{T}_{\mathcal{B}}$ (TRUE BY DEFINITION),

$U_\alpha \in \mathcal{T}_{\mathcal{B}}\ \forall\, \alpha \in \mathcal{A} \Rightarrow \bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}_{\mathcal{B}}$ (OBVIOUS) AND

$U_1, \dots, U_k \in \mathcal{T}_{\mathcal{B}} \Rightarrow U_1 \cap \dots \cap U_k \in \mathcal{T}_{\mathcal{B}}$.

I WILL PROVE THIS LAST ONE FOR $k = 2$ AND LET YOU WRITE OUT THE SIMPLE INDUCTION TO GET IT FOR ARBITRARY k (<u>EXERCISE</u> 30)

ASSUME $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$

NOTE THAT, IF $U_1 \cap U_2 = \phi$, THEN $U_1 \cap U_2 \in \mathcal{T}_{\mathcal{B}}$.

IF $U_1 \cap U_2 \neq \phi$ IT WILL SUFFICE TO FIND, FOR EACH $x \in U_1 \cap U_2$, A $U \in \mathcal{B}$ WITH $x \in U \subseteq U_1 \cap U_2$.

WRITE U_1 AND U_2 AS UNIONS OF ELEMENTS OF $\mathcal{B}$:

$$U_1 = \bigcup_\alpha V_\alpha \text{ AND } U_2 = \bigcup_\beta W_\beta$$

THEN

$$U_1 \cap U_2 = \left(\bigcup_\alpha V_\alpha\right) \cap \left(\bigcup_\beta W_\beta\right)$$

SO FOR SOME α_0 AND SOME β_0, $x \in V_{\alpha_0} \cap W_{\beta_0}$. THUS, $\exists\, U \in \mathcal{B}$ WITH $x \in U \subseteq V_{\alpha_0} \cap W_{\beta_0} \subseteq U_1 \cap U_2$. $\square$

NOW WE HAVE A MACHINE FOR MANUFACTURING TOPOLOGIES ON SETS AND WILL APPLY IT TO THE FOLLOWING SITUATION :

LET $X_1, \ldots, X_n$ BE A FINITE SET OF SPACES WITH TOPOLOGIES $\mathcal{T}_1, \ldots, \mathcal{T}_n$, RESPECTIVELY.

CONSIDER THE CARTESIAN PRODUCT <u>SET</u>

$$X = X_1 \times \cdots \times X_n = \{ (x^1, \ldots, x^n) : x^i \in X_i , i = 1, \ldots, n \}.$$

WE WANT TO USE $\mathcal{T}_1, \ldots, \mathcal{T}_n$ TO DEFINE A TOPOLOGY $\mathcal{T}$ ON X.

LET $\mathcal{B}$ DENOTE THE COLLECTION OF ALL SUBSETS OF X OF THE FORM

$$U_1 \times \cdots \times U_n$$

WHERE

$$U_i \in \mathcal{T}_i \quad , \quad i = 1, \ldots, n .$$

WE SHOW THAT $\mathcal{B}$ SATISFIES THE CONDITION IN THE PREVIOUS THEOREM. LET

$$V = V_1 \times \cdots \times V_n \in \mathcal{B}$$
$$W = W_1 \times \cdots \times W_n \in \mathcal{B}$$

AND

$$x = (x^1, \ldots, x^n) \in V \cap W$$

THEN $x^i \in V_i \cap W_i$ FOR EACH $i = 1, \ldots, n$. MOREOVER, EACH $V_i \cap W_i$ IS OPEN IN X_i SO

$$U = (V_1 \cap W_1) \times \cdots \times (V_n \cap W_n) \in \mathcal{B}.$$

THUS,

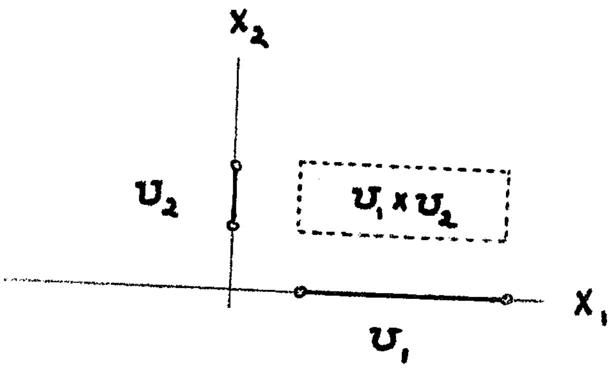
$$x \in U = (V_1 \cap W_1) \times \cdots \times (V_n \cap W_n)$$
$$\subseteq (V_1 \times \cdots \times V_n) \cap (W_1 \times \cdots \times W_n) = V \cap W$$

AS REQUIRED.

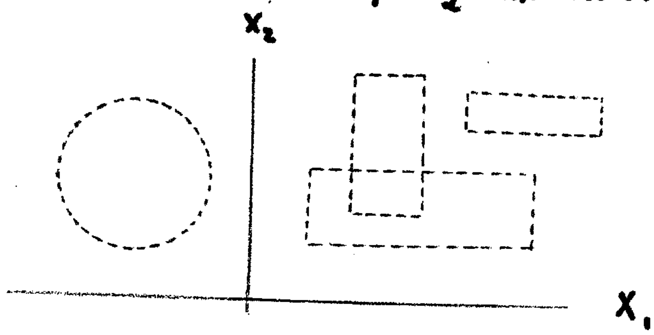
SINCE $\phi = \phi \times \cdots \times \phi$ AND $X = X_1 \times \cdots \times X_n$ ARE ALSO IN $\mathcal{B}$, $\mathcal{B}$ IS A BASIS FOR A TOPOLOGY ON X, CALLED THE PRODUCT TOPOLOGY.

PICTURES:

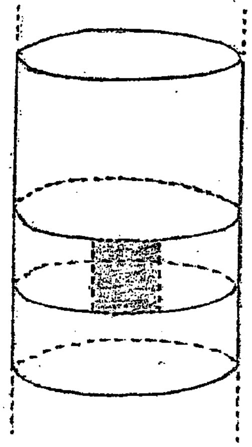
1. BASIC OPEN SETS IN $X_1 \times X_2$:



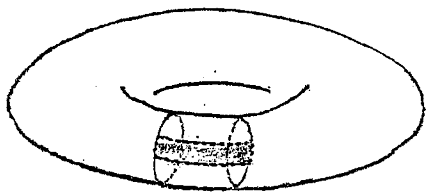
ARBITRARY OPEN SETS IN $X_1 \times X_2$ ARE UNIONS OF THESE, E.G.,



2. THE CYLINDER $S^1 \times \mathbb{R}$. TYPICAL BASIC OPEN SET IS $U \times V$, WHERE U IS OPEN IN S^1 AND V IS OPEN IN $\mathbb{R}$, E.G.,



3. THE TORUS $S^1 \times S^1$.



<u>NOTE</u>: SINCE $S^1 \subseteq \mathbb{R}^2$, $S^1 \times \mathbb{R} \subseteq \mathbb{R}^2 \times \mathbb{R} \cong \mathbb{R}^3$ AND $S^1 \times S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4$ (ALTHOUGH WE HAVE PICTURED THE TORUS IN $\mathbb{R}^3$ HERE). THUS, $S^1 \times \mathbb{R}$ AND $S^1 \times S^1$ ALSO HAVE TOPOLOGIES AS SUBSPACES OF $\mathbb{R}^3$ AND $\mathbb{R}^4$, RESPECTIVELY, AND WE WILL SHOW NOW THAT THESE ARE THE SAME AS THE PRODUCT TOPOLOGIES INTRODUCED ABOVE (I.E., THE SPACES ARE HOMEOMORPHIC).

I WILL LEAVE IT TO YOU TO SHOW THAT $\mathbb{R}^n$ WITH ITS USUAL EUCLIDEAN TOPOLOGY IS HOMEOMORPHIC TO THE PRODUCT SPACE $\mathbb{R} \times \overset{n}{\cdots} \times \mathbb{R}$ (OPEN BALLS AND OPEN RECTANGLES GIVE THE SAME TOPOLOGY). THIS IS <u>EXERCISE</u> 31.

THE RESULT WE WANT THEN FOLLOWS FROM THE

<u>THEOREM</u> : LET X_i BE A SUBSPACE OF Y_i FOR $i = 1, \dots, n$. THEN THE PRODUCT TOPOLOGY ON $X_1 \times \cdots \times X_n$ COINCIDES WITH THE RELATIVE TOPOLOGY THAT THE SUBSET $X_1 \times \cdots \times X_n$ INHERITS FROM THE PRODUCT TOPOLOGY ON $Y_1 \times \cdots \times Y_n$.

<u>PROOF</u> :

1. OPEN IN THE PRODUCT SPACE $X_1 \times \cdots \times X_n \Rightarrow$ OPEN IN THE RELATIVE TOPOLOGY

   ENOUGH TO SHOW THAT A BASIC OPEN SET $U_1 \times \cdots \times U_n$ IN THE PRODUCT IS OPEN IN THE SUBSPACE.

   U_i OPEN IN $X_i \Rightarrow U_i = X_i \cap U_i'$ WHERE U_i' OPEN IN Y_i

   $$\begin{aligned} U_1 \times \cdots \times U_n &= (X_1 \cap U_1') \times \cdots \times (X_n \cap U_n') \\ &= (X_1 \times \cdots \times X_n) \cap \underbrace{(U_1' \times \cdots \times U_n')}_{\text{OPEN IN } Y_1 \times \cdots \times Y_n} \end{aligned}$$

   THUS, $U_1 \times \cdots \times U_n$ IS OPEN IN THE RELATIVE TOPOLOGY.

2. OPEN IN THE RELATIVE TOPOLOGY $\Rightarrow$ OPEN IN THE PRODUCT $X_1 \times \cdots \times X_n$

LET $U \subseteq X_1 \times \cdots \times X_n$ BE OPEN IN THE RELATIVE TOPOLOGY.
THEN $U = (X_1 \times \cdots \times X_n) \cap U'$ WITH U' OPEN IN $Y_1 \times \cdots \times Y_n$.
$U = \phi \Rightarrow$ IT'S OPEN IN THE PRODUCT SO ASSUME $U \neq \phi$.
ENOUGH TO SHOW THAT $\forall\, x = (x^1, \ldots, x^n) \in U$ $\exists$ BASIC OPEN SET $U_1 \times \cdots \times U_n$ IN THE PRODUCT WITH
$x \in U_1 \times \cdots \times U_n \subseteq U$.

$x \in U \Rightarrow x \in U'$ SO $\exists$ BASIC OPEN SET $U_1' \times \cdots \times U_n'$ IN $Y_1 \times \cdots \times Y_n$ WITH $x \in U_1' \times \cdots \times U_n' \subseteq U'$. LETTING
$U_i = X_i \cap U_i'$,

$$\begin{aligned} x \in U_1 \times \cdots \times U_n &= (X_1 \cap U_1') \times \cdots \times (X_n \cap U_n') \\ &= (X_1 \times \cdots \times X_n) \cap (U_1' \times \cdots \times U_n') \\ &\subseteq (X_1 \times \cdots \times X_n) \cap U' = U \end{aligned}$$

AS REQUIRED. $\square$

THUS, FOR EXAMPLE, IT DOESN'T MATTER WHETHER WE THINK OF THE <u>n-TORUS</u> $S^1 \times \overset{n}{\cdots} \times S^1$ AS A PRODUCT OF n CIRCLES OR A SUBSPACE OF $\mathbb{R}^2 \times \overset{n}{\cdots} \times \mathbb{R}^2 \cong \mathbb{R}^{2n}$.

I WILL LEAVE IT TO YOU TO SHOW THAT A PRODUCT $X_1 \times \cdots \times X_n$ OF HAUSDORFF SPACES IS AGAIN HAUSDORFF (<u>EXERCISE</u> 32).

PROJECTIONS : LET $X_1 \times \cdots \times X_n$ BE A PRODUCT SPACE. FOR EACH $i = 1, \ldots, n$ DEFINE

$$\pi^i : X_1 \times \cdots \times X_n \to X_i$$

BY

$$\pi^i(x) = \pi^i(x^1, \ldots, x^n) = x^i$$

THEOREM : $\pi^i : X_1 \times \cdots \times X_n \to X_i$ IS A CONTINUOUS, OPEN MAPPING OF $X_1 \times \cdots \times X_n$ ONTO X_i.

PROOF : π^i OBVIOUSLY MAPS ONTO X_i. IT IS CONTINUOUS BECAUSE, IF U_i IS OPEN IN X_i,

$$(\pi^i)^{-1}(U_i) = X_1 \times \cdots \times U_i \times \cdots \times X_n$$

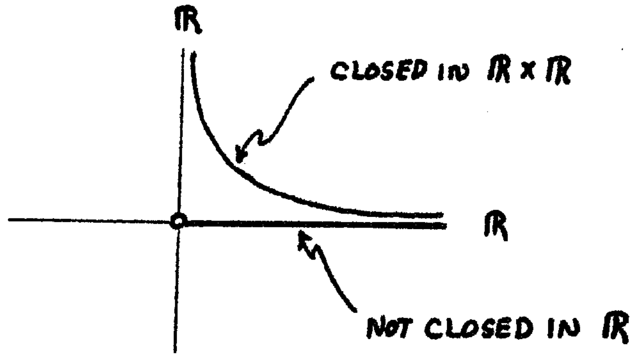
AND THIS IS OPEN IN $X_1 \times \cdots \times X_i \times \cdots \times X_n$. TO SEE THAT IT IS AN OPEN MAP, LET $U \subseteq X_1 \times \cdots \times X_n$ BE AN OPEN SET. IF $U = \phi$, THEN $\pi^i(U) = \phi$ SO IT IS OPEN IN X_i. SUPPOSE THEN THAT $U \neq \phi$. THEN $\pi^i(U) \neq \phi$ AND WE SHOW THAT IT MUST BE OPEN IN X_i.

LET $p^i \in \pi^i(U)$ BE ARBITRARY. THERE IS A $(p^1, \ldots, p^i, \ldots, p^n)$ IN U WITH i^{TH} COORDINATE p^i. U IS OPEN SO THERE IS A BASIC OPEN SET $U_1 \times \cdots \times U_i \times \cdots \times U_n$ WITH

$$(p^1, \ldots, p^i, \ldots, p^n) \in U_1 \times \cdots \times U_i \times \cdots \times U_n \subseteq U.$$

THUS, U_i IS OPEN IN X_i WITH $p^i \in U_i \subseteq \pi^i(U)$ SO $\pi^i(U)$ IS OPEN IN X_i $\square$

NOTE : π^i IS GENERALLY NOT A CLOSED MAPPING, HOWEVER, E.G.,



IF $X_1 \times \cdots \times X_n$ IS A PRODUCT SPACE, X IS SOME TOPOLOGICAL SPACE AND

$$f : X \longrightarrow X_1 \times \cdots \times X_n$$

THEN

$$\pi^i \circ f = f^i : X \to X_i$$

IS CALLED THE i^{TH} COORDINATE FUNCTION OF f. THUS, $\forall x \in X$

$$f(x) = (f^1(x), \ldots, f^n(x))$$

THEOREM : $f : X \to X_1 \times \cdots \times X_n$ IS CONTINUOUS IFF $f^i : X \to X_i$ IS CONTINUOUS $\forall\, i = 1, \ldots, n$.

PROOF : $\Rightarrow$ BECAUSE COMPOSITIONS OF CONTINUOUS MAPS ARE CONTINUOUS.

FOR $\Leftarrow$ ASSUME THAT EACH f^i IS CONTINUOUS. SINCE

$$f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$$

IT IS ENOUGH TO SHOW THAT THE INVERSE IMAGE UNDER f OF ANY BASIC OPEN SET IN $X_1 \times \cdots \times X_n$ IS OPEN IN X. BUT

$$U_1 \times \cdots \times U_n = (U_1 \times X_2 \times \cdots \times X_n) \cap \cdots \cap (X_1 \times \cdots \times X_{n-1} \times U_n)$$
$$= (\pi^1)^{-1}(U_1) \cap \cdots \cap (\pi^n)^{-1}(U_n)$$

SO

$$f^{-1}(U_1 \times \cdots \times U_n) = f^{-1}((\pi^1)^{-1}(U_1)) \cap \cdots \cap f^{-1}((\pi^n)^{-1}(U_n))$$
$$= (\pi^1 \circ f)^{-1}(U_1) \cap \cdots \cap (\pi^n \circ f)^{-1}(U_n)$$
$$= (f^1)^{-1}(U_1) \cap \cdots \cap (f^n)^{-1}(U_n)$$

AND THIS IS, INDEED, OPEN IN X BY CONTINUITY OF $f^1, \ldots, f^n$. $\square$

<u>EXERCISE</u> 33 : DEFINE A MAP $f : \mathbb{R} \to S^1 \times S^1$ BY

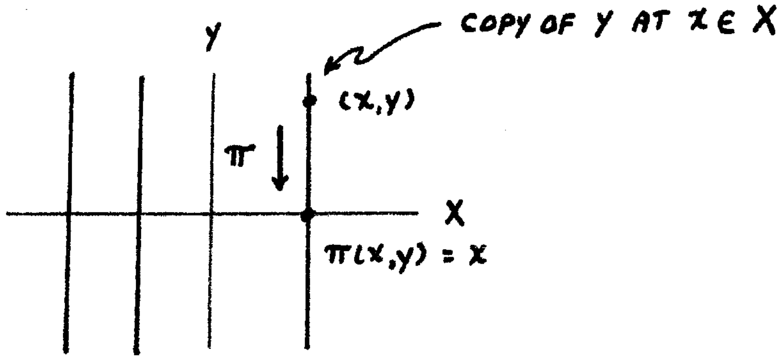
$$f(t) = (e^{it}, e^{i\alpha t})$$

WHERE α IS SOME IRRATIONAL NUMBER. THINKING OF $S^1 \times S^1$ AS A SUBSPACE OF $\mathbb{R}^4$, THIS CAN BE WRITTEN

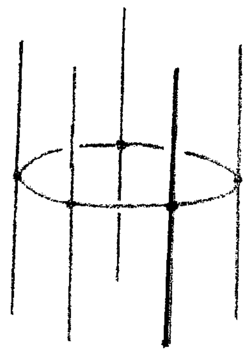
$$f(t) = (\cos t, \sin t, \cos \alpha t, \sin \alpha t)$$

SO IT IS CONTINUOUS BECAUSE EACH COORDINATE FUNCTION $f^i : \mathbb{R} \to \mathbb{R}$ IS CONTINUOUS ($f^1(t) = \cos t$, $f^2(t) = \sin t$, $f^3(t) = \cos \alpha t$, $f^4(t) = \sin \alpha t$). THUS, f IS A CONTINUOUS CURVE ON $S^1 \times S^1$. SHOW THAT f IS ONE-TO-ONE (THE CURVE DOES NOT INTERSECT ITSELF), BUT THAT ITS IMAGE IS "DENSE", I.E., INTERSECTS <u>EVERY</u> OPEN SET IN $S^1 \times S^1$. CAN YOU PICTURE IT ?

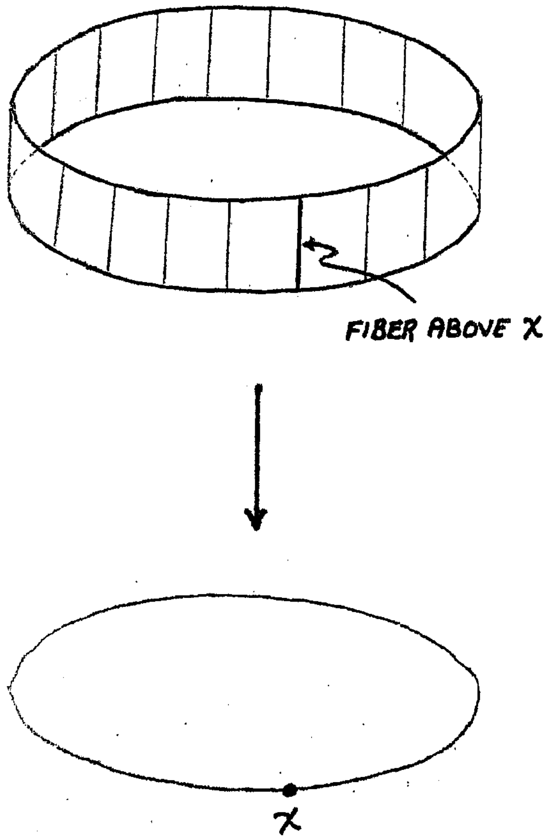
IT'S POSSIBLE TO THINK OF A PRODUCT SPACE $X \times Y$ AS A FAMILY OF COPIES OF Y PARAMETRIZED BY THE POINTS OF X WITH THE PROJECTION $\pi : X \times Y \to X$ AS PICKING OUT THE "PARAMETER VALUE" FOR EACH POINT (x, y) IN $X \times Y$.

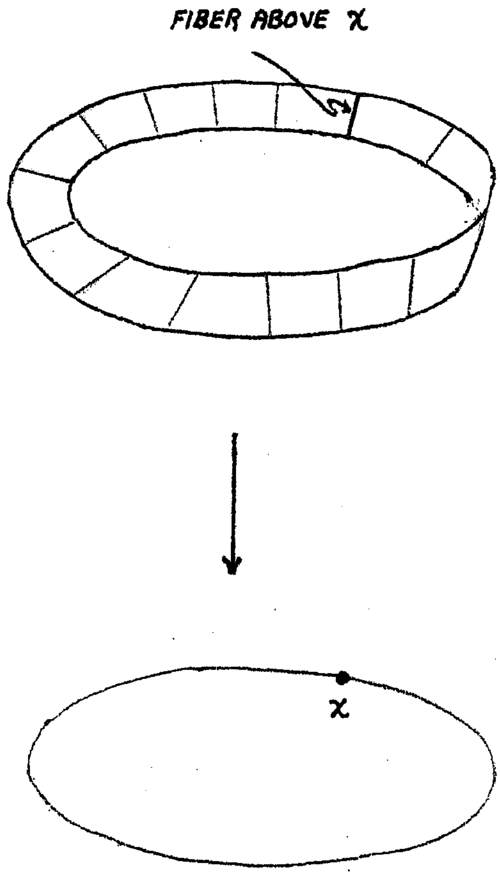


E.G., THE CYLINDER $S^1 \times \mathbb{R}$ IS A FAMILY OF LINES PARAMETRIZED BY THE POINTS OF A CIRCLE



AMONG THE MOST IMPORTANT OBJECTS IN MODERN TOPOLOGY AND GEOMETRY ARE THE SO-CALLED "FIBER BUNDLES" WHICH CONSIST OF A SPACE P, A SPACE X AND A CONTINUOUS MAP $\pi : P \to X$ OF P ONTO X WHICH LOCALLY LOOKS LIKE ONE OF THESE PRODUCTS. WE WILL NOT GET INTO THIS SERIOUSLY (YET), BUT I'LL SHOW YOU SOME PICTURES.



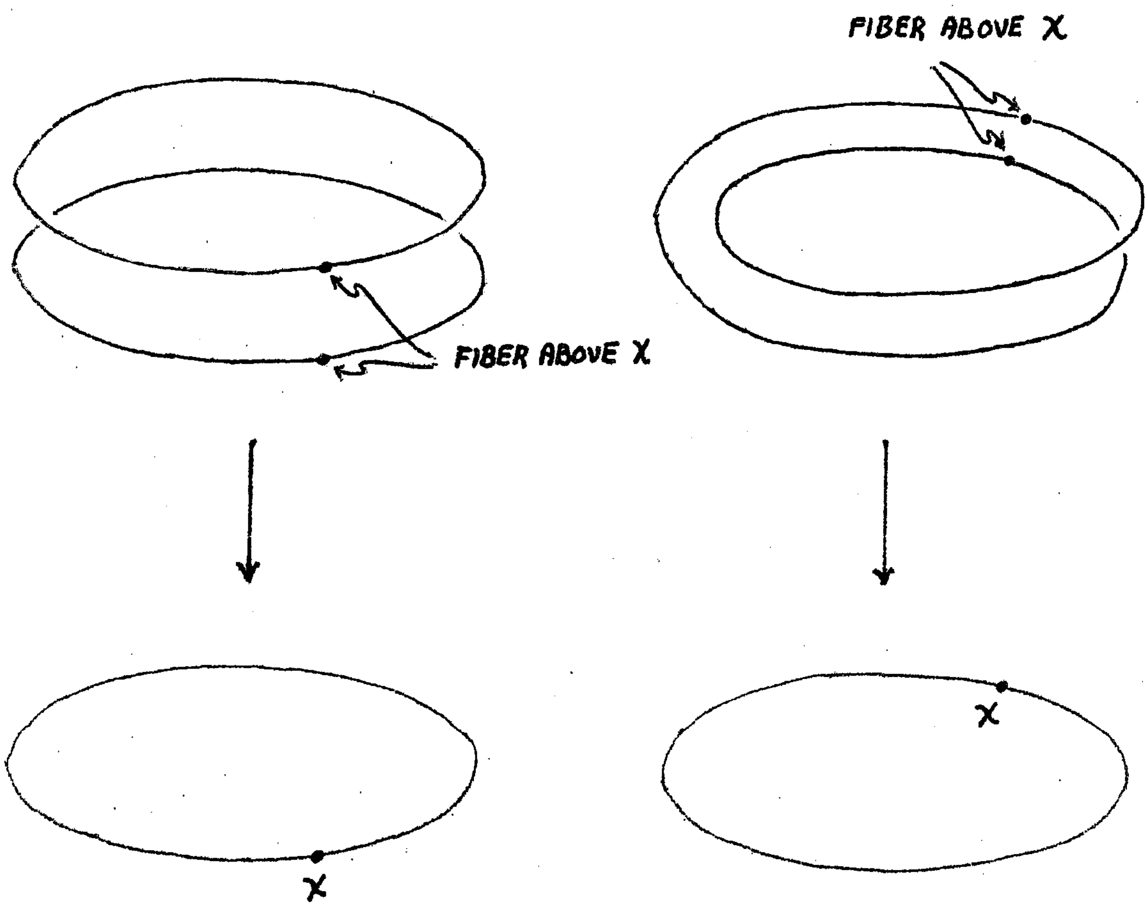


CYLINDER

A BUNDLE OF LINES
OVER THE CIRCLE

MÖBIUS STRIP

A TWISTED BUNDLE OF LINES
OVER THE CIRCLE

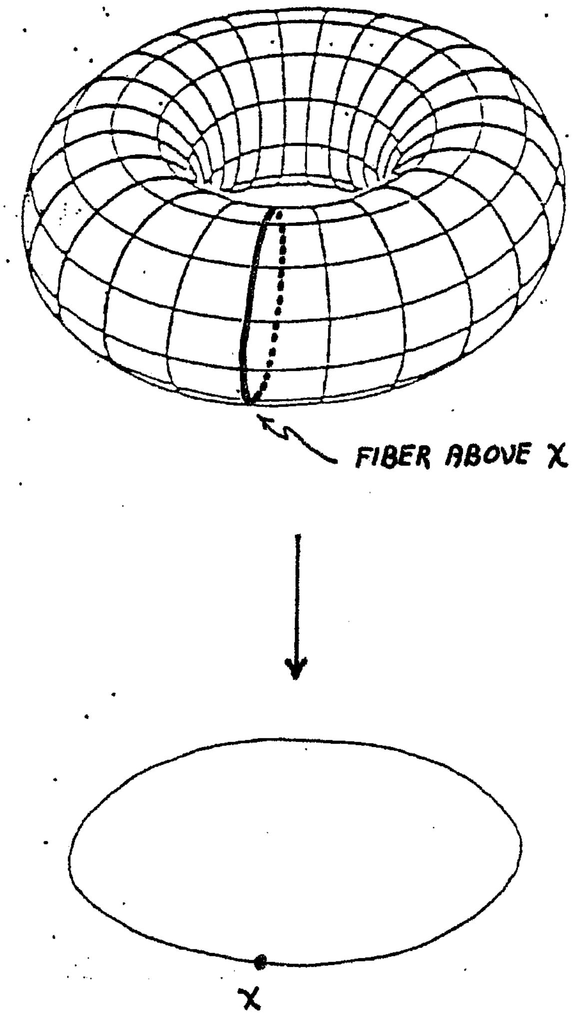


DISJOINT UNION OF TWO CIRCLES

CIRCLE

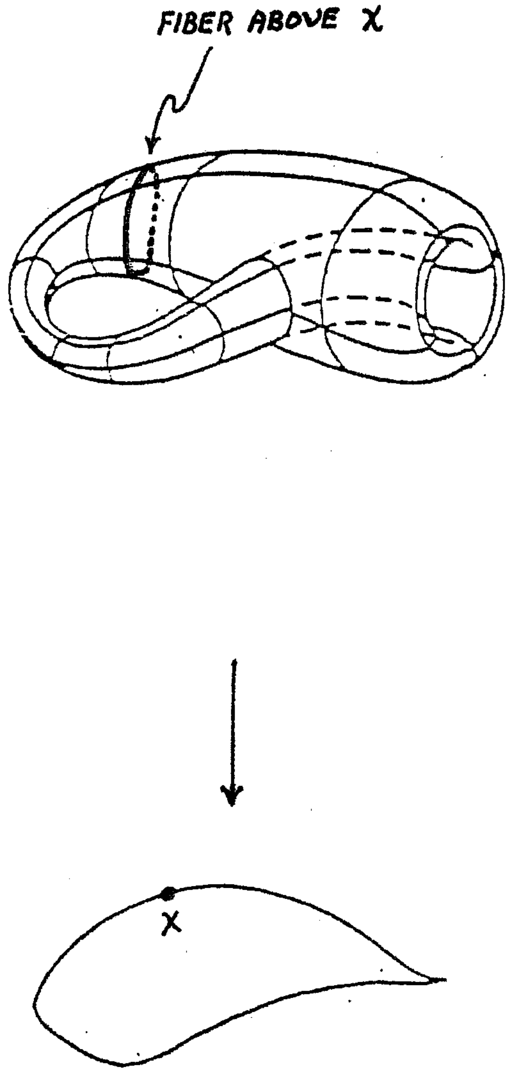
A BUNDLE OF 2-POINT SPACES OVER THE CIRCLE

A TWISTED BUNDLE OF 2-POINT SPACES OVER THE CIRCLE



TORUS

A BUNDLE OF CIRCLES OVER THE CIRCLE

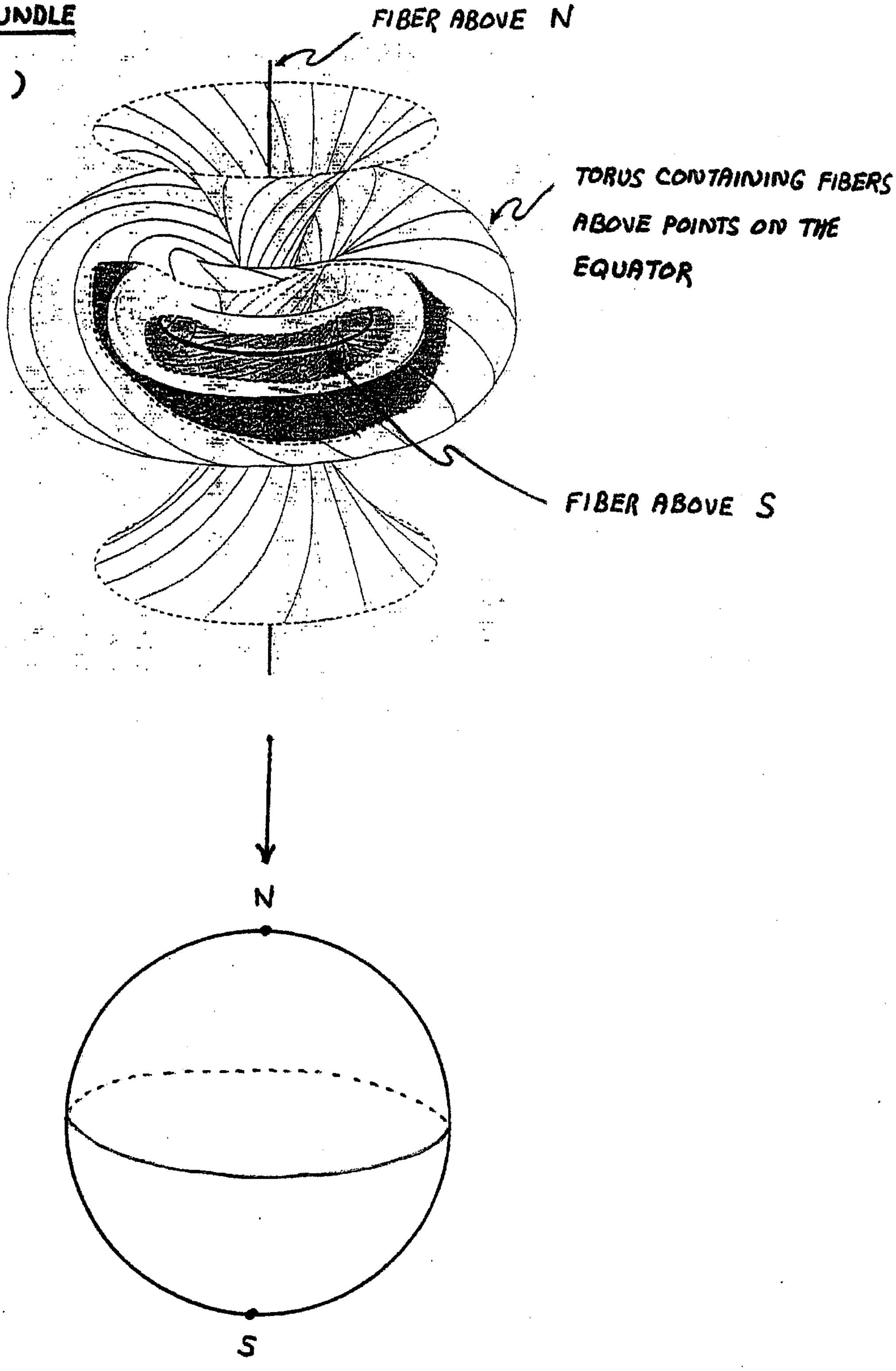


KLEIN BOTTLE

A TWISTED BUNDLE OF CIRCLES OVER THE CIRCLE

COMPLEX HOPF BUNDLE

(HOPF FIBRATION)



3 - SPHERE S^3

A TWISTED BUNDLE OF CIRCLES OVER THE 2 - SPHERE S^2

$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ IS A TOPOLOGICAL SUBSPACE OF $\mathbb{R}^2$. IT IS ALSO A GROUP UNDER COMPLEX MULTIPLICATION.

$$(x_1,y_1)\cdot(x_2,y_2) = (x_1x_2-y_1y_2\ ,\ x_1y_2+x_2y_1)$$

$$(x,y)^{-1} = (x,-y)$$

MULTIPLICATION IS A MAP

$$S^1 \times S^1 \longrightarrow S^1$$

AND INVERSION IS A MAP

$$S^1 \longrightarrow S^1$$

IF $S^1 \times S^1$ IS GIVEN THE PRODUCT TOPOLOGY, BOTH OF THESE MAPS ARE CONTINUOUS (WHY ?)

A <u>TOPOLOGICAL GROUP</u> IS A HAUSDORFF TOPOLOGICAL SPACE G THAT IS ALSO A GROUP FOR WHICH THE OPERATIONS OF MULTIPLICATION

$$(x,y) \longrightarrow xy \quad : G \times G \longrightarrow G$$

AND INVERSION

$$x \longrightarrow x^{-1} \quad : G \longrightarrow G$$

ARE CONTINUOUS.

EXAMPLES :

1. S^1 UNDER COMPLEX MULTIPLICATION AND INVERSION

2. $GL(n, \mathbb{R})$ UNDER MATRIX MULTIPLICATION AND INVERSION

EXERCISE 34 : PROVE THIS.

3. ANY SUBGROUP OF A TOPOLOGICAL GROUP WITH ITS RELATIVE TOPOLOGY IS A TOPOLOGICAL GROUP.

EXERCISE 35 : PROVE THIS.

E.G., $O(n)$ AND $SO(n)$ ARE TOPOLOGICAL GROUPS.

4. $SU(2)$ = SET OF ALL 2×2 COMPLEX MATRICES A THAT ARE UNITARY $(\bar{A}^T A = A \bar{A}^T = I)$ AND HAVE DET $A = 1$.

FIRST NOTE THAT $SU(2)$ IS A GROUP UNDER MATRIX MULTIPLICATION AND INVERSION :

$A, B \in SU(2)$: $\bar{A}^T A = I$ AND $\bar{B}^T B = I \Rightarrow$

$$(\overline{AB})^T (AB) = (\bar{A}\bar{B})^T (AB) = \bar{B}^T \bar{A}^T A B$$
$$= \bar{B}^T I B = \bar{B}^T B = I$$

DET $A = 1$ AND DET $B = 1$ $\Rightarrow$

$$\det(AB) = (\det A)(\det B) = 1.$$

NEXT WE'LL DERIVE AN EXPLICIT DESCRIPTION OF ALL OF THE ELEMENTS OF $SU(2)$:

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$$

$$\det A = 1 \Rightarrow \alpha\delta - \beta\gamma = 1$$

$$A\bar{A}^T = I \Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} \alpha\bar{\alpha} + \beta\bar{\beta} &= 1 \\ \alpha\bar{\gamma} + \beta\bar{\delta} &= 0 \\ \gamma\bar{\alpha} + \delta\bar{\beta} &= 0 \\ \gamma\bar{\gamma} + \delta\bar{\delta} &= 1 \end{aligned}$$

COMBINING A FEW OF THESE GIVES

$$\begin{aligned} \alpha\bar{\gamma} = -\beta\bar{\delta} &\Rightarrow \alpha(\bar{\gamma}\gamma) = (-\beta\gamma)\bar{\delta} \\ &\Rightarrow \alpha(1 - \delta\bar{\delta}) = (1 - \alpha\delta)\bar{\delta} \\ &\Rightarrow \alpha = \bar{\delta} \qquad (\text{SO } \delta = \bar{\alpha}) \end{aligned}$$

AND SIMILARLY

$$\gamma = -\bar{\beta}.$$

THUS,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

AND $\alpha\delta - \beta\gamma = 1$ BECOMES $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$, OR $|\alpha|^2 + |\beta|^2 = 1$.

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

WRITE $\alpha = x^1 + i x^2$ AND $\beta = x^3 + i x^4$.

$$SU(2) = \left\{ \begin{pmatrix} x^1 + i x^2 & x^3 + i x^4 \\ -x^3 + i x^4 & x^1 - i x^2 \end{pmatrix} : (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \right\}$$

WRITTEN IN THIS WAY THERE IS AN OBVIOUS BIJECTION FROM $SU(2)$ ONTO

$$\{ (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \}$$

I.E., ONTO S^3.

WE TAKE THE TOPOLOGY ON $SU(2)$ TO BE THE ONE FOR WHICH THIS BIJECTION IS A HOMEOMORPHISM, I.E., WE "TRANSFER" THE TOPOLOGY OF S^3 TO $SU(2)$.

<u>EXERCISE</u> 36: SHOW THAT $SU(2)$ IS A TOPOLOGICAL GROUP.

<u>HINT</u>: USE THE BIJECTION ABOVE TO IDENTIFY $SU(2)$ WITH S^3 AND WRITE MATRIX MULTIPLICATION AND INVERSION ON $SU(2)$ AS MAPS ON $S^3 \times S^3$ AND S^3, RESPECTIVELY.

EXERCISE 37 : LET G BE A GROUP THAT IS ALSO A HAUSDORFF TOPOLOGICAL SPACE. SHOW THAT G IS A TOPOLOGICAL GROUP IF AND ONLY IF THE MAP

$$(x,y) \longrightarrow x^{-1}y \quad : G \times G \to G$$

IS CONTINUOUS.

EXERCISE 38 : LET G BE A TOPOLOGICAL GROUP. SHOW THAT EACH OF THE FOLLOWING MAPS IS A HOMEOMORPHISM OF G ONTO G :

(a) INVERSION : $x \longrightarrow x^{-1}$

(b) FOR ANY FIXED $g \in G$, LEFT MULTIPLICATION

$$L_g : G \to G$$
$$L_g(x) = gx$$

AND RIGHT MULTIPLICATION

$$R_g : G \to G$$
$$R_g(x) = xg$$