PROOF OF EQUIVARIANT LOCALIZATION (ISOLATED ZEROS) I

BEGIN WITH A LEMMA THAT EXPLAINS WHY ONE MIGHT BE LED TO EXPECT LOCALIZATION TO THE ZERO SET.

LEMMA : M IS A COMPACT n-MANIFOLD. G IS A COMPACT LIE GROUP ACTING ON M. $\alpha \in \Omega_G^*(M)$ IS G-EQUIVARIANTLY CLOSED. THEN $\forall \xi \in \mathfrak{g}$, $\alpha(\xi)_{[n]}$ IS (DE RHAM) EXACT ON $M - Z(\xi^*)$.

" AWAY FROM THE ZERO SET OF ξ^*

$\alpha(\xi)_{[n]}$ IS COHOMOLOGICALLY TRIVIAL."

PROOF : FIX A (NONZERO) $\xi \in \mathfrak{g}$.

WRITE

$$d_{\xi^*} = d - \iota_{\xi^*}$$

THEN d_{ξ^*} ACTS ON $\Omega^*(M)$ AND SATISFIES

$$\begin{aligned} d_{\xi^*}^2 \varphi &= d_{\xi^*}(d\varphi - \iota_{\xi^*}\varphi) \\ &= d(d\varphi) - d(\iota_{\xi^*}\varphi) - \iota_{\xi^*}(d\varphi) + \iota_{\xi^*}(\iota_{\xi^*}\varphi) \\ &= -(d \circ \iota_{\xi^*} + \iota_{\xi^*} \circ d)\varphi \\ &= -\mathcal{L}_{\xi^*}\varphi \end{aligned}$$

THUS, ON

$$\Omega^*_{\xi^*}(M) = \{ \varphi \in \Omega^*(M) : \mathcal{L}_{\xi^*}\varphi = 0 \}$$

WE HAVE

$$d_{\xi^*}^2 = 0.$$

G-INVARIANCE OF α IMPLIES $\alpha(\xi) \in \Omega^*_{\xi^*}(M)$ AND, BY HYPOTHESIS,

$$d_{\xi^*}(\alpha(\xi)) = 0.$$

<u>NOTE</u>: THE REMAINDER OF THE PROOF WILL DEPEND <u>ONLY</u> ON $\mathcal{L}_{\xi^*}(\alpha(\xi)) = 0$ AND $d_{\xi^*}(\alpha(\xi)) = 0$

CHOOSE A G-INVARIANT RIEMANNIAN METRIC $\langle\ ,\ \rangle_G$ ON M AND USE IT TO CONSTRUCT A 1-FORM θ ON M DUAL TO ξ^*:

$$\theta(V) = \langle \xi^*, V \rangle_G \qquad \forall\, V \in T(TM)$$

ONE THEN SHOWS THAT

$$\mathcal{L}_{\xi^*}\theta = 0$$

SO $\theta \in \Omega^*_{\xi^*}(M)$ AND CONSEQUENTLY

$$d_{\xi^*}^2\theta = 0$$

NEXT NOTE THAT

$$d_{\xi^*}\theta = d\theta - \iota_{\xi^*}\theta = d\theta - \theta(\xi^*) = d\theta - \langle \xi^*, \xi^* \rangle_G$$

$$d_{\xi^*}\theta = -\|\xi^*\|^2 + d\theta$$

$d_{\xi^*}\theta$ IS THEREFORE A NONHOMOGENEOUS ELEMENT OF $\Omega^*(M)$ WHOSE SCALAR ($\Omega^0(M)$-) PART IS NONZERO ON $M - Z(\xi^*)$.

IT FOLLOWS THAT, ON $M - Z(\xi^*)$, $d_{\xi^*}\theta$ HAS A MULTIPLICATIVE INVERSE (RELATIVE TO $\wedge$) GIVEN FORMALLY BY THE GEOMETRIC SERIES $\left((a + A)^{-1} = \frac{1}{a} \sum_{k=0}^{\infty} (-\frac{A}{a})^k \right)$:

$$(d_{\xi^*}\theta)^{-1} = (-\|\xi^*\|^2 + d\theta)^{-1}$$
$$= -\|\xi\|^{-2}(1 + \|\xi^*\|^{-2} d\theta)$$

DEFINE, ON $M - Z(\xi^*)$,

$$\beta = \theta \wedge (d_{\xi^*}\theta)^{-1}$$

AND COMPUTE TO VERIFY THAT

$$\mathcal{L}_{\xi^*}\beta = 0$$

AND

$$d_{\xi^*}\beta = 1$$

FINALLY, DEFINE (STILL ON $M - Z(\xi^*)$)

$$\lambda = \beta \wedge \alpha(\xi) = (\theta \wedge (d_{\xi^*}\theta)^{-1}) \wedge \alpha(\xi)$$

AND COMPUTE SOME MORE TO VERIFY

$$d_{\xi^*} \lambda = \alpha(\xi)$$

SO

$$d\lambda - \iota_{\xi^*} \lambda = \alpha(\xi)$$

LOOKING AT THE TOP (n^{TH}) RANK PARTS ON BOTH SIDES AND REALIZING THAT $\iota_{\xi^*}\lambda$ HAS NONE YIELDS

$$\alpha(\xi)_{[n]} = d\lambda_{[n-1]}$$

ON $M - Z(\xi^*)$ AS REQUIRED. $\square$

NOW WE PROCEED WITH THE PROOF OF THE THEOREM.

M = COMPACT, ORIENTED, SMOOTH $2k$-MANIFOLD

G = COMPACT LIE GROUP (LIE ALGEBRA $\mathfrak{g}$) ACTING ON M ON THE LEFT AND ORIENTATION PRESERVING

$\xi \in \mathfrak{g}$ HAS $Z(\xi^*)$ FINITE

$\alpha : \mathfrak{g} \to \Omega^*(M)$ SUCH THAT $\mathcal{L}_{\xi^*}(\alpha(\xi)) = 0$ AND $d_{\xi^*}(\alpha(\xi)) = 0$

SHOW THAT

$$\int_M \alpha(\xi) = \sum_{p \in Z(\xi^*)} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{Pf(L_p(\xi))}$$

IF $Z(\xi^*) = \phi$, THEN THE RIGHT-HAND SIDE IS VACUOUSLY ZERO AND THE LEFT-HAND SIDE IS ZERO BY THE LEMMA AND STOKES' THEOREM.

NOW SUPPOSE $Z(\xi^*) \neq \phi$.

$Z(\xi^*)$ COINCIDES WITH THE FIXED POINT SET M^T OF THE TORUS ACTION ON M OBTAINED BY RESTRICTING THE G-ACTION TO

$$T = \text{CLOSURE}_G \{ \exp(-t\xi) : t \in \mathbb{R} \}$$

AND, FOR EACH p, THE T-ACTION DETERMINES THE RIGHT-HAND SIDE OF THE EQUALITY WE ARE TRYING TO PROVE. THUS, WE CAN ASSUME $G = T$ AND THEREFORE $Z(\xi^*) = M^G$.

WE HAVE SEEN THAT, FOR EACH $p \in Z(\xi^*)$, THERE IS A BASIS $\{e_1, \ldots, e_{2k}\}$ FOR $T_p(M)$, ORTHONORMAL WITH RESPECT TO A G-INVARIANT RIEMANNIAN METRIC $\langle\, , \rangle_G$ ON M AND ORIENTED, RELATIVE TO WHICH THE MATRIX OF $L_p(\xi)$ IS OF THE FORM

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & \lambda_k \\ & & & -\lambda_k & 0 \end{pmatrix}$$

WHERE $\lambda_1, \ldots, \lambda_k$ ARE NONZERO REAL NUMBERS.

IF $v \in T_p(M)$ AND WE WRITE $v = v^i e_i$, THEN

$$L_p(\xi)(v) = \lambda_1 (v^2 e_1 - v^1 e_2) + \cdots + \lambda_k (v^{2k} e_{2k-1} - v^{2k-1} e_{2k}).$$

LET $\exp_p$ BE THE $\langle\, ,\rangle_G$ - EXPONENTIAL MAP AT p. WE THEN OBTAIN NORMAL COORDINATES $x^1, \ldots, x^{2k}$ ON A NEIGHBORHOOD U_p OF p IN M. IN THESE COORDINATES

$$\xi^* | U_p = \lambda_1 \left(x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}\right) + \cdots + \lambda_k \left(x^{2k} \frac{\partial}{\partial x^{2k-1}} - x^{2k-1} \frac{\partial}{\partial x^{2k}}\right)$$

(RECALL THAT $(\exp_p)_{*v} \left(\frac{d}{dt} (\exp(L_p(-t\xi))(v)) \Big|_{t=0} \right) = \xi^*(\exp_p(v))$)

SINCE p IS A FIXED POINT OF THE ACTION, U_p CAN BE CHOSEN G-INVARIANT (RESTRICT TO SOME ε-BALL RELATIVE TO $\langle\, ,\rangle_G$).

ON U_p DEFINE A 1-FORM Θ^p BY

$$\Theta^p = \lambda_1^{-1} (x^2 dx^1 - x^1 dx^2) + \cdots + \lambda_k^{-1} (x^{2k} dx^{2k-1} - x^{2k-1} dx^{2k}).$$

THEN

$$\Theta^p (\xi^* | U_p) = (x^1)^2 + \cdots + (x^{2k})^2$$

AND

$$\mathcal{L}_{\xi^* | U_p} \Theta^p = 0.$$

NOW WE HAVE A FINITE, G-INVARIANT OPEN COVER

$$\{ U_p \}_{p \in Z(\xi^*)} \cup \{ M - Z(\xi^*) \}$$

OF M (FOR THE LAST SET KEEP IN MIND THAT $Z(\xi^*) = M^G$ SO IT, AND THEREFORE ITS COMPLEMENT, IS G-INVARIANT).

CHOOSE A G-INVARIANT PARTITION OF UNITY SUBORDINATE TO IT. WITH THIS AND THE 1-FORMS

$$\theta^p \text{ ON } U_p$$

$$\theta^0 = \langle \xi^*, \cdot \rangle_G \quad \text{ON } M - Z(\xi^*)$$

PRODUCE A 1-FORM θ ON M AND SHOW THAT

$$\mathcal{L}_{\xi^*} \theta = 0$$

AND

$$d_{\xi^*} \theta \text{ INVERTIBLE ON } M - Z(\xi^*).$$

NOW, JUST AS IN THE PROOF OF THE LEMMA, IT FOLLOWS THAT

$$\alpha(\xi) = d_{\xi^*} \left((\theta \wedge (d_{\xi^*} \theta)^{-1}) \wedge \alpha(\xi) \right)$$

ON $M - Z(\xi^*)$.

NOW WE CAN PROCEED TO THE EVALUATION OF THE INTEGRAL.

CHOOSE $\varepsilon > 0$ SUFFICIENTLY SMALL THAT THE COORDINATE BALLS

$$B_\varepsilon(p) = \{ x = (x^1, \ldots, x^{2k}) \in U_p : \|x\|^2 = (x^1)^2 + \cdots + (x^{2k})^2 \le \varepsilon \}$$

$(p \in Z(\xi^\#))$ ARE DISJOINT AND $\Theta \,|\, B_\varepsilon(p) = \Theta^p$. LET

$$S_\varepsilon(p) = \{ x \in U_p : \|x\|^2 = 1 \}.$$

SINCE $Z(\xi^\#)$ IS FINITE AND SO OF MEASURE ZERO IN M,

$$\int_M \alpha(\xi) = \int_{M - Z(\xi^\#)} \alpha(\xi) = \lim_{\varepsilon \to 0} \int_{M - \bigcup_p B_\varepsilon(p)} \alpha(\xi)$$

$$= \lim_{\varepsilon \to 0} \int_{M - \bigcup_p B_\varepsilon(p)} d_{\xi^\#} \left((\Theta \wedge (d_{\xi^\#} \Theta)^{-1}) \wedge \alpha(\xi) \right)$$

$$= \lim_{\varepsilon \to 0} \int_{M - \bigcup_p B_\varepsilon(p)} d \left((\Theta \wedge (d_{\xi^\#} \Theta)^{-1}) \wedge \alpha(\xi) \right)$$

$$= \lim_{\varepsilon \to 0} \left(- \sum_p \int_{S_\varepsilon(p)} (\Theta \wedge (d_{\xi^\#} \Theta)^{-1}) \wedge \alpha(\xi) \right)$$

(MINUS SIGN BECAUSE WE WISH TO SWITCH FROM THE EXTERIOR TO THE INTERIOR ORIENTATION ON $S_\varepsilon(p)$)

$$= \sum_p \lim_{\varepsilon \to 0} \left(- \int_{S_\varepsilon(p)} (\Theta \wedge (d_{\xi^\#} \Theta)^{-1}) \wedge \alpha(\xi) \right)$$

FOR EACH p THE CHANGE OF VARIABLE

$$x^i \longrightarrow \sqrt{\varepsilon}\, x^i \quad , \; i = 1, \dots, 2k$$

ON U_p YIELDS

$$\int_{S_\varepsilon(p)} (\Theta \wedge (d_{\xi *}\Theta)^{-1}) \wedge \alpha(\xi) = \int_{S_1(p)} (\Theta \wedge (d_{\xi *}\Theta)^{-1}) \wedge \alpha_\varepsilon(\xi)$$

WHERE

$$\alpha_\varepsilon(\xi)(x^1, \dots, x^{2k}, dx^1, \dots, dx^{2k}) =$$

$$\alpha(\xi)(\sqrt{\varepsilon}\, x^1, \dots, \sqrt{\varepsilon}\, x^{2k}, \sqrt{\varepsilon}\, dx^1, \dots, \sqrt{\varepsilon}\, dx^{2k})$$

($\Theta \wedge (d_{\xi *}\Theta)^{-1}$ IS INVARIANT). THUS,

$$\lim_{\varepsilon \to 0} \left(- \int_{S_\varepsilon(p)} (\Theta \wedge (d_{\xi *}\Theta)^{-1}) \wedge \alpha(\xi) \right) =$$

$$\underbrace{\left(- \int_{S_1(p)} \Theta \wedge (d_{\xi *}\Theta)^{-1} \right)}\, \alpha(\xi)_{[0]}(p)$$

FINISH THE PROOF BY SHOWING THAT THIS EQUALS

$$\frac{(-2\pi)^k}{Pf(L_p(\xi))}$$

FOR EACH $p \in Z(\xi^*)$.

JUST COMPUTE

$$-\int_{S_1(p)} \theta \wedge (d_{\xi *}\theta)^{-1} = -\int_{S_1(p)} \theta \wedge (-1 + d\theta)^{-1}$$

BECAUSE $\iota_{\xi *}\theta^{p} = (x^1)^2 + \cdots + (x^{2k})^2 = 1$ ON $S_1(p)$

$$= \int_{S_1(p)} \theta \wedge (1 - d\theta)^{-1} = \int_{S_1(p)} \theta \wedge (1 + d\theta + (d\theta)^2 + \cdots + (d\theta)^k)$$

$$= \int_{S_1(p)} \theta \wedge (d\theta)^{k-1}$$ BECAUSE $\dim S_1(p) = 2k-1$

$$= \int_{B_1(p)} (d\theta)^k$$ BY STOKES' THEOREM BECAUSE $d(\theta \wedge (d\theta)^{k-1}) = d\theta \wedge (d\theta)^{k-1} - \theta \wedge d((d\theta)^{k-1}) = (d\theta)^k$ AND BECAUSE WE HAVE ALREADY SWITCHED TO THE INTERIOR ORIENTATION ON $S_1(p)$

$$= \int_{B_1(p)} (d\theta^{p})^k = \int_{B_1(p)} ((-2)(\lambda_1^{-1} dx^1 \wedge dx^2 + \cdots + \lambda_k^{-1} dx^{2k-1} \wedge dx^{2k}))^k$$

$$= (-2)^k k! (\lambda_1 \cdots \lambda_k)^{-1} \int_{B_1(p)} dx^1 \wedge \cdots \wedge dx^k$$

$$= \frac{(-2)^k k! \left(\frac{\pi^k}{k!}\right)}{Pf(L_p(\xi))}$$

$$= \frac{(-2\pi)^k}{Pf(L_p(\xi))}$$