PROOF OF EQUIVARIANT LOCALIZATION (ISOLATED ZEROS) II

LEMMA : LET G BE A COMPACT LIE GROUP THAT ACTS SMOOTHLY ON THE LEFT ON THE n-MANIFOLD M. LET $\alpha \in \Omega_G^*(M)$ BE G-EQUIVARIANTLY CLOSED. FOR EACH $\xi \in \mathfrak{g}$ WRITE $\alpha(\xi) = \alpha(\xi)_{[0]} + \alpha(\xi)_{[1]} + \cdots + \alpha(\xi)_{[n]}$, WHERE $\alpha(\xi)_{[i]}$ IS HOMOGENEOUS OF DEGREE i. THEN $\alpha(\xi)_{[n]}$ IS EXACT ON $M - Z(\xi^\#)$.

"AWAY FROM THE ZERO SET OF $\xi^\#$,

$\alpha(\xi)_{[n]}$ IS COHOMOLOGICALLY TRIVIAL."

PROOF : THE RESULT IS VACUOUS IF $\xi = 0$ SO FIX SOME NONZERO $\xi \in \mathfrak{g}$.

REMARKS : BEFORE PROCEEDING WITH THE PROOF WE RECORD SOME GENERAL OBSERVATIONS. FOR ANY $\alpha \in \Omega_G^*(M)$ AND ANY $\xi \in \mathfrak{g}$, $\alpha(\xi) \in \Omega^*(M)$ AND $(d_G\alpha)(\xi) = (d - \iota_{\xi^\#})(\alpha(\xi))$. IF ξ IS HELD FIXED THEN

$$d_{\xi^\#} := d - \iota_{\xi^\#}$$

ACTS ON $\Omega^*(M)$ AND $\forall\, \varphi \in \Omega^*(M)$,

$$\begin{aligned} d_{\xi^\#}^2 \varphi &= d_{\xi^\#}(d\varphi - \iota_{\xi^\#}\varphi) = d(d\varphi - \iota_{\xi^\#}\varphi) - \iota_{\xi^\#}(d\varphi - \iota_{\xi^\#}\varphi) \\ &= -(d \circ \iota_{\xi^\#} + \iota_{\xi^\#} \circ d)\varphi \\ &= -\mathcal{L}_{\xi^\#}\varphi \qquad \text{(BY THE CARTAN FORMULA)} \end{aligned}$$

THUS, ON THE SUBALGEBRA

$$\Omega^*_{\xi^\#}(M) = \{ \varphi \in \Omega^*(M) : \mathcal{L}_{\xi^\#} \varphi = 0 \}$$

OF $\xi^\#$ - INVARIANT FORMS WE HAVE

$$d_{\xi^\#}^2 = 0.$$

APPLYING ANALOGOUS FORMULAS FOR d AND $\iota_{\xi^\#}$ ONE OBTAINS A LEIBNITZ RULE FOR $d_{\xi^\#}$:

$$(1) \qquad d_{\xi^\#}(\omega \wedge \eta) = (d_{\xi^\#}\omega) \wedge \eta + [\omega_{[0]} - \omega_{[1]} + \cdots + (-1)^n \omega_{[n]}] \wedge d_{\xi^\#} \eta$$

NOTE : THE REMAINDER OF THE PROOF WILL RELY ONLY ON $d_{\xi^\#}(\alpha(\xi)) = 0$ AND THESE PROPERTIES OF $d_{\xi^\#}$. IN PARTICULAR, THE CONCLUSION WILL ALSO BE TRUE OF ANY $\Omega^*(M)$ - VALUED MAP $\xi \longrightarrow \alpha(\xi)$ ON $\mathfrak{g}$, EVEN IF IT IS NOT POLYNOMIAL IN ξ, PROVIDED $\mathcal{L}_{\xi^\#}(\alpha(\xi)) = 0$ AND $d_{\xi^\#}(\alpha(\xi)) = 0$.

ALSO NOTE : IN THE PHYSICS LITERATURE ONE OFTEN FINDS " EQUIVARIANT COHOMOLOGY " DEFINED (RELATIVE TO SOME KILLING VECTOR FIELD V FOR SOME RIEMANNIAN MANIFOLD M) AS THE COHOMOLOGY OF

$$\Omega^*_V(M) = \{ \varphi \in \Omega^*(M) : \mathcal{L}_V \varphi = 0 \}$$

$$d_V = d - \iota_V \quad (\text{OR } d + \iota_V , \text{ OR } d + s\,\iota_V , \ s \in \mathbb{C})$$

THEN d_V CARRIES EVEN/ODD FORMS TO ODD/EVEN FORMS, I.E., BOSONS/FERMIONS TO FERMIONS/BOSONS AND CAN BE VIEWED AS A "SUPERSYMMETRY OPERATOR".

NOW WE RETURN TO THE PROOF. USE THE G-INVARIANT METRIC $\langle\, ,\, \rangle_G$ ON M TO CONSTRUCT A 1-FORM θ DUAL TO ξ^* :

$$\theta(V) = \langle \xi^*, V \rangle_G$$

FOR ALL VECTOR FIELDS V ON M. WE CLAIM THAT θ IS ξ^*-INVARIANT, I.E., THAT

(2) $$\mathcal{L}_{\xi^*}\theta = 0$$

TO SEE THIS WE FIX $p \in M$ AND $v_p \in T_p(M)$ AND SHOW THAT

$$(\mathcal{L}_{\xi^*}\theta)_p(v_p) = 0.$$

BY DEFINITION, $\mathcal{L}_{\xi^*}\theta = \frac{d}{dt}(L_{\exp(-t\xi)}{}^*\theta)\big|_{t=0}$ SO

$$(\mathcal{L}_{\xi^*}\theta)_p(v_p) = \frac{d}{dt}\left(\theta_{\exp(-t\xi)\cdot p}((L_{\exp(-t\xi)})_{*p}(v_p))\right)\Big|_{t=0}$$

(3) $$(\mathcal{L}_{\xi^*}\theta)_p(v_p) = \frac{d}{dt}\langle \xi^*(\exp(-t\xi)\cdot p), (L_{\exp(-t\xi)})_{*p}(v_p)\rangle_G \Big|_{t=0}$$

NEXT WE NOTE THAT

(4) $$\xi^*(\exp(-t\xi)\cdot p) = (L_{\exp(-t\xi)})_{*p}(\xi^*(p))$$

TO SEE THIS WE REGARD $\xi^*(p) = \frac{d}{ds}(L_{\exp(-s\xi)}(p))\big|_{s=0}$ AS THE VELOCITY VECTOR AT $s=0$ OF THE CURVE $\gamma(s) = L_{\exp(-s\xi)}(p) = \exp(-s\xi)\cdot p$. THEN

$$(L_{\exp(-t\xi)})_{*p}(\xi^{\#}(p)) = (L_{\exp(-t\xi)})_{*p}(\gamma'(0)) = (L_{\exp(-t\xi)} \circ \gamma)'(0).$$

BUT

$$L_{\exp(-t\xi)} \circ \gamma(s) = \exp(-t\xi)\cdot(\exp(-s\xi)\cdot p)$$
$$= \exp(-(t+s)\xi)\cdot p$$

SO

$$\gamma'(0) = \frac{d}{ds}(\exp(-(t+s)\xi)\cdot p)\Big|_{s=0}$$
$$= \frac{d}{ds}(\exp(-s\xi)\cdot(\exp(-t\xi)\cdot p))\Big|_{s=0}$$
$$= \frac{d}{ds}(L_{\exp(-s\xi)}(\exp(-t\xi)\cdot p))\Big|_{s=0}$$
$$= \xi^{\#}(\exp(-t\xi)\cdot p)$$

THIS PROVES (4) AND WITH IT (3) BECOMES

$$(\mathcal{L}_{\xi^{\#}}\theta)_p(v_p) = \frac{d}{dt}\langle (L_{\exp(-t\xi)})_{*p}(\xi^{\#}(p)), (L_{\exp(-t\xi)})_{*p}(v_p)\rangle_G\Big|_{t=0}$$
$$= \frac{d}{dt}\langle \xi^{\#}(p), v_p\rangle_G\Big|_{t=0}$$
$$= 0$$

BECAUSE $L_{\exp(-t\xi)}$ IS AN ISOMETRY OF $\langle\ ,\ \rangle_G$. THIS PROVES (2).

THUS, WE HAVE $\theta \in \Omega^{*}_{\xi^{\#}}(M)$ SO IT FOLLOWS THAT

(5) $$d_{\xi^{\#}}^{2}\theta = 0$$

NEXT NOTICE THAT

$$d_{\xi^\#}\theta = d\theta - \iota_{\xi^\#}\theta = d\theta - \theta(\xi^\#) = d\theta - \langle \xi^\#, \xi^\# \rangle_G$$

$$d_{\xi^\#}\theta = -\|\xi^\#\|^2 + d\theta$$

SO $d_{\xi^\#}\theta$ <u>IS A NONHOMOGENEOUS ELEMENT OF</u> $\Omega^*(M)$ <u>WHOSE SCALAR</u> ($\Omega^0(M)$-) <u>PART IS</u> $-\|\xi^\#\|^2$ <u>AND IS THEREFORE NONZERO ON</u> $M - Z(\xi^\#)$.

<u>REMARK</u>: A NONHOMOGENEOUS ELEMENT OF $\Omega^*(M)$ WITH NONVANISHING SCALAR PART ALWAYS HAS A MULTIPLICATIVE INVERSE (RELATIVE TO $\wedge$). WE GIVE A PROOF OF THIS. WRITE SUCH AN ELEMENT AS $a + A$ WHERE $a \in \Omega^0(M)$ IS NONVANISHING AND $A_{[0]} = 0$. WE CLAIM THAT

$$(a+A)^{-1} = \frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{A}{a}\right)^k \qquad \text{(A FINITE SUM)}$$

(MOTIVATION: $\frac{1}{x+y} = \frac{1}{x}\,\frac{1}{1-(-\frac{y}{x})} = \frac{1}{x}\sum_{k=0}^{\infty}\left(-\frac{y}{x}\right)^k$ PROVIDED $x \neq 0$ AND $\left|-\frac{y}{x}\right| < 1$). TO PROVE THIS SUPPOSE $\dim M = n$ AND COMPUTE, E.G.,

$$(a+A) \wedge \left(\frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{A}{a}\right)^k\right) = \frac{1}{a}(a+A)\wedge\left(1 - \frac{A}{a} + \frac{A^2}{a^2} - \cdots + (-1)^n\frac{A^n}{a^n}\right)$$

$$= \frac{1}{a}\Big[\, a - A + \frac{A^2}{a} - \cdots + (-1)^n\frac{A^n}{a^{n-1}} + A - \frac{A^2}{a} + \cdots + (-1)^{n-1}\frac{A^n}{a^{n-1}} + \underbrace{(-1)^n\frac{A^{n+1}}{a^n}}_{0}\,\Big]$$

$$= \frac{1}{a}[\,a\,] = 1.$$

WE CONCLUDE THAT, ON $M - Z(\xi^*)$, $d_{\xi^*}\theta = -\|\xi^*\|^2 + d\theta$ IS INVERTIBLE AND

$$(d_{\xi^*}\theta)^{-1} = (-\|\xi^*\|^2 + d\theta)^{-1}$$
$$= -\|\xi^*\|^{-2}\,(1 + \|\xi^*\|^{-2} d\theta)$$

SO, ON $M - Z(\xi^*)$, WE CAN DEFINE

$$\beta = \theta \wedge (d_{\xi^*}\theta)^{-1}$$

(WHICH ONE OFTEN SEES WRITTEN AS $\dfrac{\theta}{d_{\xi^*}\theta}$). WE CLAIM THAT

(6) $$d_{\xi^*}\beta = 1$$

AND

(7) $$\mathcal{L}_{\xi^*}\beta = 0.$$

TO SEE THESE WE COMPUTE

$$d_{\xi^*}\beta = d_{\xi^*}(\theta \wedge (d_{\xi^*}\theta)^{-1})$$
$$= (d_{\xi^*}\theta) \wedge (d_{\xi^*}\theta)^{-1} + [\theta_{[0]} - \theta_{[1]} + \theta_{[2]} - \cdots] \wedge d_{\xi^*}((d_{\xi^*}\theta)^{-1})$$
$$= \underline{1} + [0 - \theta + 0 - \cdots] \wedge d_{\xi^*}((d_{\xi^*}\theta)^{-1})$$
$$= \underline{1} - \theta \wedge d_{\xi^*}((d_{\xi^*}\theta)^{-1})$$

AND

$$\mathcal{L}_{\xi^*}\beta = \mathcal{L}_{\xi^*}(\theta \wedge (d_{\xi^*}\theta)^{-1})$$
$$= \underbrace{(\mathcal{L}_{\xi^*}\theta)}_{0} \wedge (d_{\xi^*}\theta)^{-1} + \theta \wedge \mathcal{L}_{\xi^*}((d_{\xi^*}\theta)^{-1})$$
$$= \theta \wedge \mathcal{L}_{\xi^*}((d_{\xi^*}\theta)^{-1}).$$

THUS, TO PROVE (6) AND (7) IT WILL SUFFICE TO SHOW THAT BOTH $d_{\xi^\#}((d_{\xi^\#}\theta)^{-1})$ AND $\mathcal{L}_{\xi^\#}((d_{\xi^\#}\theta)^{-1})$ ARE ZERO. THESE ARE BOTH PROVED IN THE SAME WAY, BEGINNING WITH

$$(d_{\xi^\#}\theta)\wedge(d_{\xi^\#}\theta)^{-1} = 1 .$$

FOR EXAMPLE, COMPUTING $d_{\xi^\#}$ OF BOTH SIDES GIVES

$$d_{\xi^\#}\left((d_{\xi^\#}\theta)\wedge(d_{\xi^\#}\theta)^{-1}\right) = 0$$

AND SO

$$\underset{\substack{\uparrow\\0}}{d_{\xi^\#}(d_{\xi^\#}\theta)}\wedge(d_{\xi^\#}\theta)^{-1} + [\underset{\substack{\uparrow\\-\|\xi^\#\|^2}}{(d_{\xi^\#}\theta)_{[0]}} - \underset{\substack{\uparrow\\0}}{(d_{\xi^\#}\theta)_{[1]}} + \underset{\substack{\uparrow\\d\theta}}{(d_{\xi^\#}\theta)_{[2]}} - \cdots]\wedge d_{\xi^\#}((d_{\xi^\#}\theta)^{-1}) = 0$$

$$(-\|\xi^\#\|^2 + d\theta)\wedge d_{\xi^\#}((d_{\xi^\#}\theta)^{-1}) = 0$$

$$(d_{\xi^\#}\theta)\wedge d_{\xi^\#}((d_{\xi^\#}\theta)^{-1}) = 0$$

NOW WEDGE BOTH SIDES ON THE LEFT BY $(d_{\xi^\#}\theta)^{-1}$ TO GET

$$d_{\xi^\#}((d_{\xi^\#}\theta)^{-1}) = 0.$$

THE PROOF FOR $\mathcal{L}_{\xi^\#}((d_{\xi^\#}\theta)^{-1})$ IS AS FOLLOWS:

$$\mathcal{L}_{\xi^\#}\left((d_{\xi^\#}\theta)^{-1}\wedge(d_{\xi^\#}\theta)\right) = 0$$

$$\mathcal{L}_{\xi^\#}((d_{\xi^\#}\theta)^{-1})\wedge(d_{\xi^\#}\theta) + (d_{\xi^\#}\theta)^{-1}\wedge\mathcal{L}_{\xi^\#}(d_{\xi^\#}\theta) = 0$$

BUT

$$
\begin{aligned}
\mathcal{L}_{\xi^\#}(d_{\xi^\#}\theta) &= \mathcal{L}_{\xi^\#}(-\|\xi^\#\|^2 + d\theta) \\
&= -\underbrace{\mathcal{L}_{\xi^\#}(\langle \xi^\#, \xi^\# \rangle_G)}_{0 \text{ BECAUSE } \langle\ ,\ \rangle_G \text{ IS } G\text{-INVARIANT}} + \mathcal{L}_{\xi^\#}(d\theta) \\
&= d(\mathcal{L}_{\xi^\#}\theta) \quad \text{BECAUSE } \mathcal{L}_{\xi^\#} \circ d = d \circ \mathcal{L}_{\xi^\#} \\
&= d(0) \\
&= 0
\end{aligned}
$$

SO

$$
\mathcal{L}_{\xi^\#}((d_{\xi^\#}\theta)^{-1}) \wedge (d_{\xi^\#}\theta) = 0 .
$$

NOW WEDGE BOTH SIDES ON THE RIGHT BY $(d_{\xi^\#}\theta)^{-1}$. WITH THIS THE PROOFS OF (6) AND (7) ARE COMPLETE.

FINALLY, WE DEFINE

$$
\lambda = \beta \wedge \alpha(\xi) = (\theta \wedge (d_{\xi^\#}\theta)^{-1}) \wedge \alpha(\xi)
$$

AND COMPUTE

$$
\begin{aligned}
d_{\xi^\#}\lambda &= d_{\xi^\#}(\beta \wedge \alpha(\xi)) \\
&= \underbrace{(d_{\xi^\#}\beta)}_{1} \wedge \alpha(\xi) + [\beta_{[0]} - \beta_{[1]} + \cdots] \wedge \underbrace{d_{\xi^\#}(\alpha(\xi))}_{0} \\
&= \alpha(\xi)
\end{aligned}
$$

THUS,

(8) $$ d\lambda - \iota_{\xi^\#}\lambda = \alpha(\xi) $$

NOW LOOK AT TOP (n^{TH}) RANK PARTS ON BOTH SIDES OF (8). $\iota_{\xi^\#}\lambda$ HAS NONE (SINCE λ CAN HAVE NO NONZERO RANK $n+1$ TERM ON AN n- MANIFOLD) SO

$$\alpha(\xi)_{[n]} = (d\lambda)_{[n]} = d\lambda_{[n-1]}$$

AS REQUIRED. $\square$

FOR FUTURE REFERENCE WE SUMMARIZE WHAT WE HAVE PROVED :

$$d_{\xi^\#}(\alpha(\xi)) = 0 \quad \text{AND} \quad \Theta := \langle \xi^\#, \cdot \rangle_G$$

$$\mathcal{L}_{\xi^\#}\Theta = 0 \quad \text{SO} \quad d_{\xi^\#}^2\Theta = 0$$

ON $M - Z(\xi^\#)$,

$$(d_{\xi^\#}\Theta)^{-1} = -\|\xi^\#\|^{-2}(1 + \|\xi^\#\|^{-2}d\Theta)$$

$$d_{\xi^\#}(\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) = 0$$

$$\mathcal{L}_{\xi^\#}(\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) = 0$$

$$d_{\xi^\#}((\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) \wedge \alpha(\xi)) = \alpha(\xi)$$

$$\alpha(\xi)_{[n]} = d\left(((\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) \wedge \alpha(\xi))_{[n-1]}\right)$$

EQUIVARIANT LOCALIZATION THEOREM (ISOLATED ZEROS): LET M BE A COMPACT, ORIENTED, SMOOTH n-MANIFOLD AND G A COMPACT LIE GROUP ACTING ON M (ON THE LEFT) BY ORIENTATION PRESERVING DIFFEOMORPHISMS. LET $\xi \in \mathfrak{g}$ BE SUCH THAT THE INFINITESIMAL ACTION $\xi^{\#} \in T(M)$ HAS ISOLATED ZEROS THEN $n = 2k$ AND, FOR ANY G-EQUIVARIANTLY CLOSED FORM α ON M,

$$(9) \qquad \int_M \alpha(\xi) = \sum_{p \in Z(\xi^{\#})} (-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{Pf(L_p(\xi))}$$

REMARK: WE EMPHASIZE AGAIN THAT, JUST AS IN THE CASE THE PREVIOUS LEMMA, THE PROOF WE GIVE DOES NOT RELY ON THE FULL STRENGTH OF THE ASSUMPTION THAT α IS A G-EQUIVARIANTLY CLOSED FORM ON M (I.E., POLYNOMIAL IN ξ), BUT ONLY ON THE FACT THAT

$$d_{\xi^{\#}}(\alpha(\xi)) = 0$$

FOR THE PARTICULAR $\xi \in \mathfrak{g}$ UNDER CONSIDERATION.

PROOF: IF $Z(\xi^{\#}) = \phi$, THEN THE RIGHT-HAND SIDE OF (9) IS VACUOUSLY ZERO AND OUR LEMMA IMPLIES THAT $\alpha(\xi)_{[n]}$ IS EXACT EVERYWHERE ON M SO STOKES' THEOREM GIVES $\int_M \alpha(\xi) = 0$ AS WELL. THUS, WE ASSUME $Z(\xi^{\#}) \neq \phi$.

WE HAVE ALREADY OBSERVED THAT THE ZERO SET $Z(\xi^{\#})$ COINCIDES WITH THE FIXED POINT SET M^T OF THE TORUS ACTION ON M OBTAINED BY RESTRICTING

THE G-ACTION TO $T = \text{CLOSURE}_G \{ \exp(-t\xi) : t \in \mathbb{R} \}$. SINCE THIS RESTRICTED ACTION COMPLETELY DETERMINES $\xi^\#$ AND THEREFORE $L_p(\xi)$ FOR EACH $p \in Z(\xi^\#) = M^T$, IT DETERMINES THE RIGHT-HAND SIDE OF (1) AND SO WE CAN ASSUME $G = T$. THE ADVANTAGE TO THIS IS THAT NOW EVERY ZERO OF $\xi^\#$ IS A FIXED POINT OF THE G-ACTION SO, AS WE OBSERVED EARLIER, WE CAN FIND, FOR EACH $p \in Z(\xi^\#)$ A G-INVARIANT NEIGHBORHOOD U_p OF p AND (NORMAL) COORDINATES $x^1, \dots, x^{2k}$ ON U_p SUCH THAT

(10) $$\xi^\# | U_p = \lambda_1 (x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}) + \dots + \lambda_k (x^{2k} \frac{\partial}{\partial x^{2k-1}} - x^{2k-1} \frac{\partial}{\partial x^{2k}})$$

AND

(11) $$Pf(L_p(\xi)) = \lambda_1 \cdots \lambda_k \neq 0.$$

ON U_p DEFINE A 1-FORM Θ^p BY

(12) $$\Theta^p = \lambda_1^{-1} (x^2 dx^1 - x^1 dx^2) + \dots + \lambda_k^{-1} (x^{2k} dx^{2k-1} - x^{2k-1} dx^{2k})$$

THEN

(13) $$\Theta^p(\xi^\# | U_p) = (x^1)^2 + (x^2)^2 + \dots + (x^{2k-1})^2 + (x^{2k})^2$$

BECAUSE, FOR EXAMPLE, $\lambda_1^{-1}(x^2 dx^1 - x^1 dx^2)(\lambda_1 x^2 \frac{\partial}{\partial x^1} - \lambda_1 x^1 \frac{\partial}{\partial x^2} + \cdots) =$
$\lambda_1^{-1}(x^2(\lambda_1 x^2) - x^1(-\lambda_1 x^1)) = (x^1)^2 + (x^2)^2$, ETC.

MOREOVER, WE CLAIM THAT

(14) $$\mathcal{L}_{\xi^\# | U_p} \Theta^p = 0$$

ON U_p. TO PROVE THIS WE USE THE CARTAN FORMULA (AND, FOR

CONVENIENCE, DROP THE " $|U_p$ ").

$$\mathcal{L}_{\xi^*}(\theta^P) = d(\iota_{\xi^*}\theta^P) + \iota_{\xi^*}(d\theta^P)$$

BUT

$$d(\iota_{\xi^*}\theta^P) = d(\theta^P(\xi^*)) = d((x^1)^2 + \cdots + (x^{2k})^2)$$
$$= 2x^1 dx^1 + \cdots + 2x^{2k} dx^{2k}$$

AND

$$\iota_{\xi^*}(d\theta^P) = \iota_{\xi^*}(\lambda_1^{-1}(dx^2 \wedge dx^1 - dx^1 \wedge dx^2) + \cdots$$
$$+ \lambda_k^{-1}(dx^{2k} \wedge dx^{2k-1} - dx^{2k-1} \wedge dx^{2k}))$$

$$= -2\lambda_1^{-1}\iota_{\xi^*}(dx^1 \wedge dx^2) - \cdots - 2\lambda_k^{-1}\iota_{\xi^*}(dx^{2k-1} \wedge dx^{2k})$$

$$= -2\lambda_1^{-1}\iota_{\xi^*}(dx^1 \otimes dx^2 - dx^2 \otimes dx^1) - \cdots$$
$$- 2\lambda_k^{-1}\iota_{\xi^*}(dx^{2k-1} \otimes dx^{2k} - dx^{2k} \otimes dx^{2k-1})$$

$$= -2\lambda_1^{-1}(dx^1(\xi^*)dx^2 - dx^2(\xi^*)dx^1) - \cdots$$
$$- 2\lambda_k^{-1}(dx^{2k-1}(\xi^*)dx^{2k} - dx^{2k}(\xi^*)dx^{2k-1})$$

$$= -2\lambda_1^{-1}((\lambda_1 x^2)dx^2 - (-\lambda_1 x^1)dx^1) - \cdots$$
$$- 2\lambda_k^{-1}((\lambda_k x^{2k})dx^{2k} - (-\lambda_k x^{2k-1})dx^{2k-1})$$

$$= -2x^1 dx^1 - 2x^2 dx^2 - \cdots - 2x^{2k-1}dx^{2k-1} - 2x^{2k}dx^{2k}$$

$$= -d(\iota_{\xi^*}\theta^P)$$

AS REQUIRED.

EACH OF THE SETS U_p, $p \in Z(\xi^\#)$, IS G-INVARIANT BY CONSTRUCTION. SINCE $Z(\xi^\#)$ IS THE FIXED POINT SET OF THE G (= T) ACTION, IT IS OBVIOUSLY G-INVARIANT AND THEREFORE SO IS $M - Z(\xi^\#)$. THUS,

$$\{U_p\}_{p \in Z(\xi^\#)} \cup \{M - Z(\xi^\#)\}$$

IS A (FINITE) G-INVARIANT OPEN COVER OF M. BY "AVERAGING OVER G" JUST AS ONE DOES TO PRODUCE G-INVARIANT RIEMANNIAN METRICS ONE CAN PRODUCE A G-INVARIANT PARTITION OF UNITY SUBORDINATE TO THIS OPEN COVER. WITH THIS AND THE 1-FORMS

$$\Theta^p \text{ on } U_p$$

$$\Theta^0 = \langle \xi^\#, \cdot \rangle_G \text{ on } M - Z(\xi^\#)$$

WE CAN PRODUCE A 1-FORM Θ ON M WITH THE FOLLOWING PROPERTIES:

1. Θ AGREES WITH Θ^p ON SOME NEIGHBORHOOD OF p $\forall p \in Z(\xi^\#)$

2. $\mathcal{L}_{\xi^\#}\Theta = 0$

<u>NOTE</u>: WE HAVE ALREADY SEEN THAT $\mathcal{L}_{\xi^\# | U_p}\Theta^p = 0$ (THIS IS (14)) AND $\mathcal{L}_{\xi^\#}\Theta^0 = 0$ (PROOF OF THE LEMMA). WE NEED G-INVARIANCE OF THE PARTITION OF UNITY TO GET

$$\begin{aligned}\mathcal{L}_{\xi^\#}\Theta &= \mathcal{L}_{\xi^\#}(\rho_0\Theta^0 + \textstyle\sum \rho_p\Theta^p) \\ &= \underbrace{(\mathcal{L}_{\xi^\#}\rho_0)}_{0}\Theta^0 + \rho_0\underbrace{\mathcal{L}_{\xi^\#}\Theta^0}_{0} + \textstyle\sum\big(\underbrace{(\mathcal{L}_{\xi^\#}\rho_p)}_{0}\Theta^p + \rho_p\underbrace{\mathcal{L}_{\xi^\#}\Theta^p}_{0}\big) \\ &= 0.\end{aligned}$$

3. $d_{\xi^\#}\Theta$ IS INVERTIBLE ON $M - Z(\xi^\#)$

NOTE : $d_{\xi^\#}\Theta = d\Theta - \iota_{\xi^\#}\Theta$ IS INVERTIBLE IF AND ONLY IF $\iota_{\xi^\#}\Theta$ IS NONVANISHING. BUT

$$\iota_{\xi^\#}\Theta = \iota_{\xi^\#}(\rho_0\Theta^0 + \sum \rho_p \Theta^p)$$
$$= \rho_0\, \iota_{\xi^\#}\Theta^0 + \sum \rho_p\, \iota_{\xi^\#}\Theta^p$$

AND WE HAVE ALREADY SEEN THAT $\iota_{\xi^\#}\Theta^0 = \langle \xi^\#, \xi^\# \rangle_G$ AND $\iota_{\xi^\#}\Theta^p = (x^1)^2 + \cdots + (x^{2k})^2$ ARE POSITIVE ON $M - Z(\xi^\#)$. SINCE EACH ρ IS ≥ 0 EVERYWHERE AND AT LEAST ONE IS NONZERO AT EACH POINT, THE RESULT FOLLOWS.

NOW WE CAN ARGUE JUST AS IN THE PROOF OF THE LEMMA THAT, ON $M - Z(\xi^\#)$, $d_{\xi^\#}(\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) = 0$, $\mathcal{L}_{\xi^\#}(\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) = 0$ AND, FROM THESE AND $d_{\xi^\#}(\alpha(\xi)) = 0$,

(15) $$d_{\xi^\#}((\Theta \wedge (d_{\xi^\#}\Theta)^{-1}) \wedge \alpha(\xi)) = \alpha(\xi)$$

WITH THIS WE CAN COMPUTE $\int_M \alpha(\xi)$ AND PROVE (9) AS FOLLOWS :

CHOOSE $\varepsilon > 0$ SUFFICIENTLY SMALL THAT THE COORDINATE BALLS

$$B_\varepsilon(p) = \{ x = (x^1, \dots, x^{2k}) : \|x\|^2 = (x^1)^2 + \cdots + (x^{2k})^2 \leq \varepsilon^2 \}$$

IN U_p ARE DISJOINT FOR THE VARIOUS $p \in Z(\xi^\#)$ AND $\Theta = \Theta^p$ ON $B_\varepsilon(p)$. LET

$$S_\varepsilon(p) = \{ x : \|x\| = \varepsilon \}$$

FOR EACH $p \in Z(\xi^*)$.

SINCE $Z(\xi^*)$ IS A FINITE SET AND SO OF MEASURE ZERO,

$$\int_M \alpha(\xi) = \int_{M - Z(\xi^*)} \alpha(\xi) = \lim_{\varepsilon \to 0} \int_{M - \bigcup_p B_\varepsilon(p)} \alpha(\xi)$$

$$= \lim_{\varepsilon \to 0} \int_{M - \bigcup_p B_\varepsilon(p)} d_{\xi^*}\left((\theta \wedge (d_{\xi^*}\theta)^{-1}) \wedge \alpha(\xi) \right) \qquad \text{BY (15)}$$

$$= \lim_{\varepsilon \to 0} \int_{M - \bigcup_p B_\varepsilon(p)} d\left((\theta \wedge (d_{\xi^*}\theta)^{-1}) \wedge \alpha(\xi) \right)$$

BECAUSE ι_{ξ^*} OF SOMETHING CAN HAVE NO TOP RANK PART

$$= \lim_{\varepsilon \to 0} \left(- \sum_p \int_{S_\varepsilon(p)} (\theta \wedge (d_{\xi^*}\theta)^{-1}) \wedge \alpha(\xi) \right)$$

BY STOKES' THEOREM. THE EXTRA MINUS SIGN HAS BEEN INCLUDED BECAUSE WE WISH TO SWITCH FROM THE EXTERIOR TO THE INTERIOR ORIENTATION ON $S_\varepsilon(p)$

$$= \sum_p \underbrace{\lim_{\varepsilon \to 0} \left(- \int_{S_\varepsilon(p)} (\theta \wedge (d_{\xi^*}\theta)^{-1}) \wedge \alpha(\xi) \right)}$$

THE PROOF WILL BE COMPLETED BY SHOWING THAT THIS EQUALS

$$(-2\pi)^k \frac{\alpha(\xi)_{[0]}(p)}{\mathrm{Pf}(L_p(\xi))}$$

CONSIDER A FIXED $B_\varepsilon(p)$. MAKE A CHANGE OF COORDINATES

$$x^i \longrightarrow \varepsilon^{\frac{1}{2}} x^i \quad , \quad i = 1, \ldots, 2k$$

ON U_p. IN THE NEW COORDINATES, $B_\varepsilon(p) \longrightarrow B_1(p)$ AND $S_\varepsilon(p) \to S_1(p)$. WRITE $\alpha_\varepsilon(\xi)$ FOR $\alpha(\xi)$ IN THESE NEW COORDINATES, I.E.,

$$\alpha_\varepsilon(\xi)(x^1, \ldots, x^{2k}, dx^1, \ldots, dx^{2k}) = \alpha(\xi)(\varepsilon^{\frac{1}{2}} x^1, \ldots, \varepsilon^{\frac{1}{2}} x^{2k}, \varepsilon^{\frac{1}{2}} dx^1, \ldots, \varepsilon^{\frac{1}{2}} dx^{2k}).$$

NOTE THAT, AS $\varepsilon \to 0$, ALL OF THE $\alpha_\varepsilon(\xi)_{[i]}$ WITH $i > 0$ APPROACH ZERO (EVEN IF $\alpha(\xi)_{[i]}$ HAS CONSTANT COMPONENTS, BECAUSE OF THE PRESENCE OF THE dx^i), WHEREAS $\alpha_\varepsilon(\xi)_{[0]} \longrightarrow \alpha(\xi)_{[0]}(p)$ BECAUSE $p = (0, \ldots, 0)$.

NOW CONSIDER $\Theta \wedge (d_{\xi *}\Theta)^{-1}$ IN THE NEW COORDINATES. NEAR p, $\Theta = \Theta^p$ SO (12) SHOWS THAT, IN THE NEW COORDINATES, Θ PICKS UP A FACTOR OF ε. FROM OUR EARLIER CALCULATIONS OF $d\Theta^p = -2\lambda_1^{-1} dx^1 \wedge dx^2 - \cdots - 2\lambda_k^{-1} dx^{2k-1} \wedge dx^{2k}$ AND $\iota_{\xi *}\Theta^p = (x^1)^2 + \cdots + (x^{2k})^2$ IT IS CLEAR THAT $d_{\xi *}\Theta = d\Theta - \iota_{\xi *}\Theta$ ALSO PICKS UP A FACTOR OF ε IN THE NEW COORDINATES AND SO $(d_{\xi *}\Theta)^{-1}$ PICKS UP A FACTOR OF $\frac{1}{\varepsilon}$. THE NET RESULT IS THAT IN $\Theta \wedge (d_{\xi *}\Theta)^{-1}$ THESE TWO FACTORS CANCEL AND $\Theta \wedge (d_{\xi *}\Theta)^{-1}$ IS INVARIANT UNDER OUR CHANGE OF COORDINATES.

THUS,

$$\int_{S_\varepsilon(p)} (\Theta \wedge (d_{\xi *}\Theta)^{-1}) \wedge \alpha(\xi) = \int_{S_1(p)} (\Theta \wedge (d_{\xi *}\Theta)^{-1}) \wedge \alpha_\varepsilon(\xi)$$

AND

$$\lim_{\varepsilon\to 0}\Big(-\int_{S_\varepsilon(p)}(\theta\wedge(d_{\xi^*}\theta)^{-1}\wedge\alpha(\xi))\Big) = \Big(-\int_{S_1(p)}\theta\wedge(d_{\xi^*}\theta)^{-1}\Big)\,\alpha(\xi)_{[0]}(p)$$

SO ALL THAT REMAINS IS TO PROVE

$$(16)\qquad -\int_{S_1(p)}\theta\wedge(d_{\xi^*}\theta)^{-1} = \frac{(-2\pi)^k}{Pf(L_p(\xi))}$$

FOR THIS WE COMPUTE

$$-\int_{S_1(p)}\theta\wedge(d_{\xi^*}\theta)^{-1} = -\int_{S_1(p)}\theta\wedge(-1+d\theta)^{-1}$$ BECAUSE $\iota_{\xi^*}\theta^P = (x^1)^2+\dots+(x^{2k})^2 = 1$ ON $S_1(p)$

$$= \int_{S_1(p)}\theta\wedge(1-d\theta)^{-1}$$

$$= \int_{S_1(p)}\theta\wedge(1+d\theta+(d\theta)^2+\dots+(d\theta)^{k-1}+(d\theta)^k)$$

$$= \int_{S_1(p)}\theta\wedge(d\theta)^{k-1}$$ SINCE $\dim S_1(p) = 2k-1$

$$= \int_{B_1(p)}(d\theta)^k$$ BY STOKES' THEOREM BECAUSE $d(\theta\wedge(d\theta)^{k-1}) = d\theta\wedge(d\theta)^{k-1} - \theta\wedge d((d\theta)^{k-1}) = (d\theta)^k - 0 = (d\theta)^k$ AND BECAUSE WE ALREADY SWITCHED TO THE INTERIOR ORIENTATION ON $S_1(p)$.

NOW, NEAR p,

$$(d\theta)^k = \left((-2)(\lambda_1^{-1} dx^1 \wedge dx^2 + \cdots + \lambda_k^{-1} dx^{2k-1} \wedge dx^{2k}) \right)^k$$

$$= (-2)^k \, k! \, (\lambda_1 \cdots \lambda_k)^{-1} dx^1 \wedge \cdots \wedge dx^{2k}$$

SO

$$\int_{S_1(p)} \theta \wedge (d_{\xi_0}\theta)^{-1} = (-2)^k \, k! \, (\lambda_1 \cdots \lambda_k)^{-1} \int_{B_1(p)} dx^1 \wedge \cdots \wedge dx^{2k}$$

$$= \frac{(-2)^k \, k! \, \left(\frac{\pi^k}{k!} \right)}{\mathrm{Pf}\,(L_p(\xi))}$$

$$= \frac{(-2\pi)^k}{\mathrm{Pf}\,(L_p(\xi))} .$$

THIS COMPLETES THE PROOF OF (16) AND THEREFORE OF THE EQUIVARIANT LOCALIZATION THEOREM. $\square$