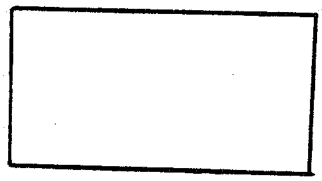
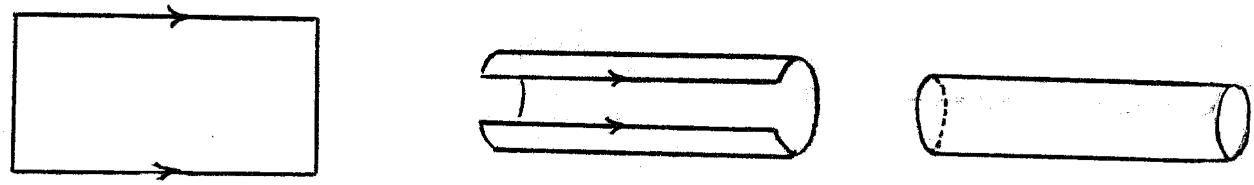
QUOTIENT SPACES

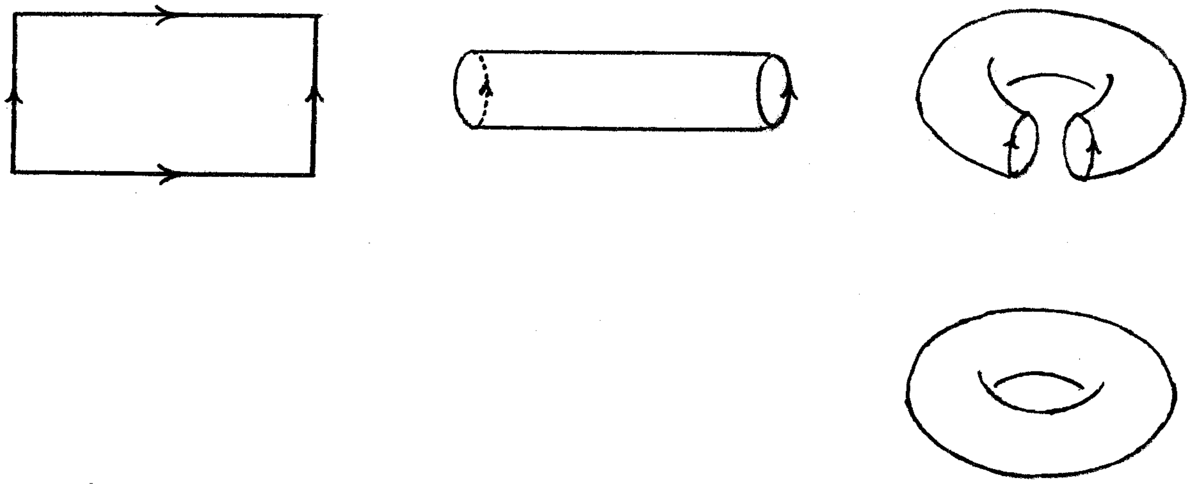
A RECTANGLE



IS A PRETTY BORING THING. BUT START GLUING ITS EDGES TOGETHER AND YOU GET SOME INTERESTING STUFF.



CYLINDER



TORUS

THE PRECISE WAY OF "GLUING" THINGS TOGETHER TOPOLOGICALLY IS A "QUOTIENT SPACE".

A FEW PRELIMINARIES :

X = A TOPOLOGICAL SPACE

Y = A SET

$\pi : X \to Y$ A MAP OF X ONTO Y

EXERCISE 19 : SHOW THAT THE COLLECTION OF ALL SUBSETS U OF Y WITH THE PROPERTY THAT $\pi^{-1}(U)$ IS OPEN IN X FORMS A TOPOLOGY FOR Y.

THIS TOPOLOGY FOR Y IS CALLED THE QUOTIENT TOPOLOGY DETERMINED BY π, π IS CALLED THE QUOTIENT MAP, AND Y IS CALLED THE QUOTIENT SPACE OF X BY π.

NOTICE THAT IF Y HAS THE QUOTIENT TOPOLOGY DETERMINED BY π, THEN π IS CLEARLY CONTINUOUS. HOWEVER, MUCH MORE IS TRUE.

LEMMA : SUPPOSE Y HAS THE QUOTIENT TOPOLOGY DETERMINED BY $\pi : X \to Y$. IF Z IS ANY TOPOLOGICAL SPACE, THEN A MAP

$$g : Y \to Z$$

IS CONTINUOUS IF AND ONLY IF ITS COMPOSITION WITH π

$$g \circ \pi : X \to Z$$

IS CONTINUOUS.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, dotted, "g\circ\pi"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

PROOF : $\Rightarrow$ IS CLEAR SINCE $\pi : X \to Y$ CONTINUOUS AND $g : Y \to Z$ CONTINUOUS $\Rightarrow$ $g \circ \pi : X \to Z$ CONTINUOUS (EXERCISE 5).

FOR $\Leftarrow$ SUPPOSE $g \circ \pi : X \to Z$ IS CONTINUOUS. LET V BE AN OPEN SET IN Z. THEN

$$(g\circ\pi)^{-1}(V) = \pi^{-1}(g^{-1}(V))$$

IS OPEN IN X. BY DEFINITION OF THE QUOTIENT TOPOLOGY, THIS MEANS THAT $g^{-1}(V)$ IS OPEN IN Y AND THIS MEANS THAT g IS CONTINUOUS. $\square$

EXERCISE 20 : LET X AND Y BE TOPOLOGICAL SPACES AND $f : X \to Y$ IS A CONTINUOUS MAP OF X ONTO Y. f IS SAID TO BE AN OPEN (RESPECTIVELY, CLOSED) MAP IF U OPEN IN $X \Rightarrow f(U)$ OPEN IN Y (RESPECTIVELY, C CLOSED IN $X \Rightarrow f(C)$ CLOSED IN Y). SHOW THAT IF f IS EITHER OPEN OR CLOSED, THEN Y HAS THE QUOTIENT TOPOLOGY DETERMINED BY $f : X \to Y$.

SO WHAT DOES THIS HAVE TO DO WITH "GLUING" THINGS TOGETHER?

LET X BE A TOPOLOGICAL SPACE AND SUPPOSE THAT THERE IS ALSO DEFINED ON X SOME EQUIVALENCE RELATION $\sim$. THEN $\forall x, y, z \in X$

(a) $x \sim x$

(b) $x \sim y \Rightarrow y \sim x$

(c) $x \sim y$ AND $y \sim z \Rightarrow x \sim z$

FOR ANY $x \in X$ THE EQUIVALENCE CLASS OF x IS DENOTED

$$[x] = \{ y \in X : y \sim x \}$$

AND THESE PARTITION X UP INTO PAIRWISE DISJOINT SUBSETS. THE COLLECTION OF SUCH EQUIVALENCE CLASSES IS DENOTED

$$X/\sim \ = \{ [x] : x \in X \}$$

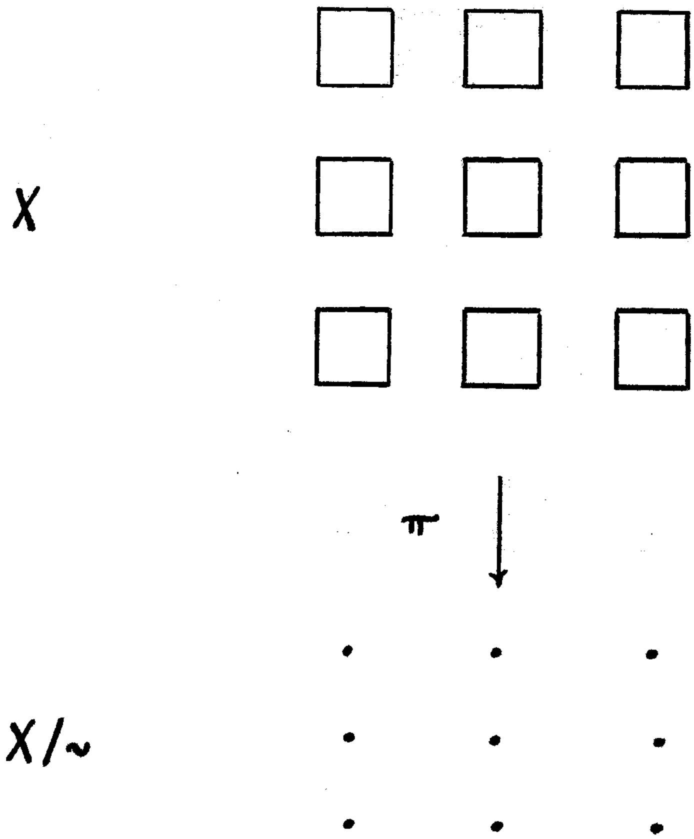
<u>NOTE</u>: IN $X/\sim$, EACH $[x]$ IS REGARDED AS A SINGLE POINT, I.E., ALL THE ELEMENTS IN THE EQUIVALENCE CLASS IN X HAVE BEEN "IDENTIFIED" ("GLUED TOGETHER").

THE MAP

$$\pi : X \to X/\sim$$

$$\pi(x) = [x]$$

IS ONTO AND IS CALLED THE QUOTIENT MAP.



NOW WE TURN $X/\sim$ INTO A TOPOLOGICAL SPACE BY GIVING IT THE QUOTIENT TOPOLOGY DETERMINED BY $\pi : X \longrightarrow X/\sim$ (A SUBSET U OF $X/\sim$ IS OPEN IFF AN OPEN SET IN X RESULTS FROM "BLOWING UP" ALL OF ITS POINTS TO SUBSETS OF X).

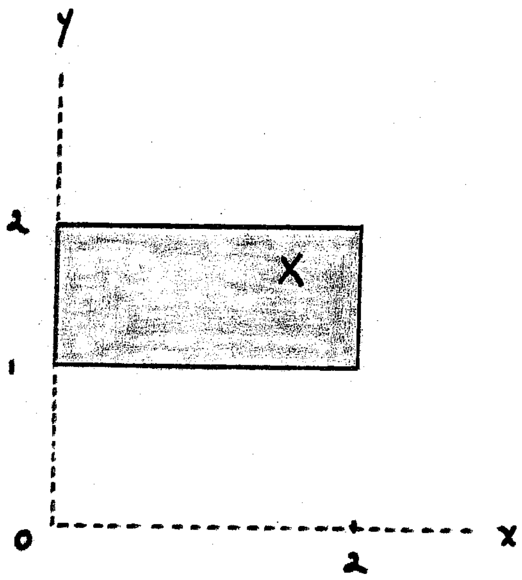
WE WILL CONSTRUCT SOME NONTRIVIAL EXAMPLES SHORTLY, BUT FIRST I WILL LIST A FEW MORE BASIC EXAMPLES TO HELP YOU GET USED TO THE IDEA.

<u>EXERCISE</u> 21 : FOR EACH OF THE FOLLOWING, "GUESS" WHAT YOU THINK THE QUOTIENT SPACE $X/\sim$ WILL BE HOMEOMORPHIC TO, THEN PICK <u>ONE</u> OF THEM AND PROVE THAT YOUR GUESS IS CORRECT.

LET

$$X = [0,2] \times [1,2] = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 1 \le y \le 2\}$$

WITH ITS SUBSPACE TOPOLOGY FROM $\mathbb{R}^2$.



(a) DEFINE $\sim$ ON X BY

$$(x',y') \sim (x,y) \quad \text{IFF} \quad x' = x$$

(b) DEFINE $\sim$ ON X BY

$(x',y') \sim (x,y)$ IFF ONE OF THE FOLLOWING IS TRUE:

(A) $x' = x$ AND $y' = y$

(B) IF $x' \neq x$, THEN EITHER

(i) $x' = 0$, $x = 2$, AND $y' = y$, OR

(ii) $x' = 2$, $x = 0$, AND $y' = y$

(c) DEFINE $\sim$ ON X BY

$(x',y') \sim (x,y)$ IFF ONE OF THE FOLLOWING IS TRUE :

(A) $x' = x$ AND $y' = y$

(B) IF $x' \neq x$, THEN EITHER

(i) $x' = 0$, $x = 2$, AND $y + y' = 3$, OR

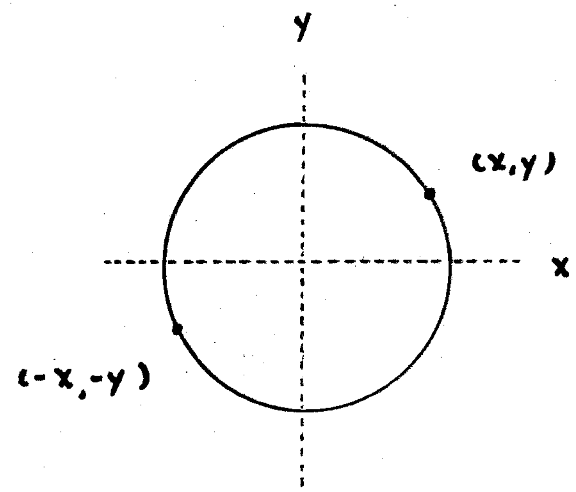
(ii) $x' = 2$, $x = 0$, AND $y + y' = 3$

(d) DEFINE $\sim$ ON $X = S^1$ (THE UNIT CIRCLE IN THE XY-PLANE) BY

$(x',y') \sim (x,y)$ IFF EITHER

(A) $x' = x$ AND $y' = y$, OR

(B) $x' = -x$ AND $y' = -y$



THE REMAINING EXAMPLES TAKE A BIT MORE WORK, BUT ARE AMONG THE MOST IMPORTANT OBJECTS IN TOPOLOGY AND GEOMETRY.

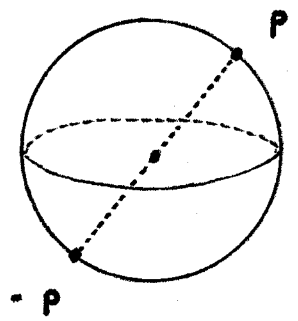
EXAMPLES :

1. (REAL PROJECTIVE PLANE $\mathbb{RP}^2$) BEGIN WITH THE 2-SPHERE

$$S^2 = \{ p = (p^1, p^2, p^3) \in \mathbb{R}^3 : \| p \| = 1 \}$$

IF $p \in S^2$, THEN ITS ANTIPODE IS

$$-p = (-p^1, -p^2, -p^3) \in S^2 .$$



" ANTIPODAL POINTS "

DEFINE AN EQUIVALENCE RELATION $\sim$ ON S^2 BY

$$q \sim p \qquad \text{IFF} \qquad q = p \ \text{ OR } \ q = -p$$

EQUIVALENCE CLASS OF p IS

$$[p] = [p^1, p^2, p^3] = \{ p, -p \}$$

SET OF ALL EQUIVALENCE CLASSES IS

$$\mathbb{RP}^2 = \{ [p] : p \in S^2 \}$$

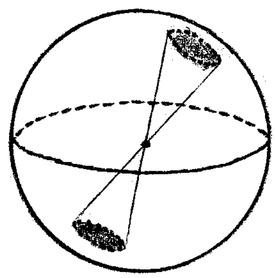
PROJECTION MAP :

$$\pi : S^2 \rightarrow \mathbb{RP}^2$$

$$\pi(p) = [p]$$

PROVIDE $\mathbb{RP}^2$ WITH THE QUOTIENT TOPOLOGY DETERMINED BY π.

$$\mathcal{U} \subseteq \mathbb{RP}^2 \text{ OPEN IFF } \pi^{-1}(\mathcal{U}) \text{ OPEN IN } S^2$$



A TYPICAL $\pi^{-1}(\mathcal{U})$ CONTAINS $-p$ FOR EACH p IN IT

THIS TOPOLOGICAL SPACE IS CALLED THE REAL PROJECTIVE PLANE.

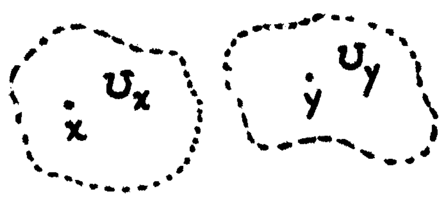
WE WILL CONSTRUCT (SOMETHING LIKE) A PICTURE OF IT SHORTLY.

FIRST WE WILL STUDY IT A BIT.

ANY TOPOLOGICAL SPACE WORTH ITS SALT HAS ENOUGH OPEN SETS TO "SEPARATE POINTS". MORE PRECISELY,

A TOPOLOGICAL SPACE X IS SAID TO BE HAUSDORFF (OR T_2) IF, FOR ANY TWO DISTINCT POINTS x AND y IN X THERE EXIST OPEN SETS $\mathcal{U}_x$ AND $\mathcal{U}_y$ IN X WITH

$$x \in \mathcal{U}_x, \quad y \in \mathcal{U}_y, \text{ AND } \mathcal{U}_x \cap \mathcal{U}_y = \phi.$$



ANY METRIC SPACE IS CLEARLY HAUSDORFF AND WE SHOW NOW THAT $\mathbb{RP}^2$ IS AS WELL.

EXERCISE 22 : SHOW THAT $\pi : S^2 \to \mathbb{RP}^2$ IS AN OPEN MAP.

HINT : FOR ANY $V \subseteq S^2$ LET $-V = \{-p : p \in V\}$ AND SHOW THAT V OPEN $\Rightarrow -V$ OPEN. THEN SHOW $\pi^{-1}(\pi(V)) = V \cup (-V)$.

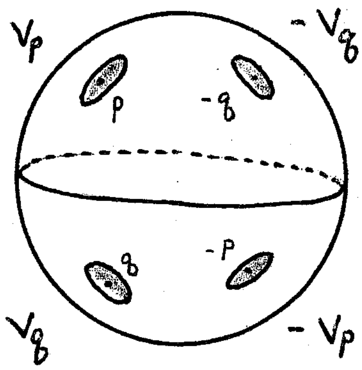
NOW LET $[p]$ AND $[q]$ BE DISTINCT POINTS OF $\mathbb{RP}^2$. THEN, IN S^2,

$$(q^1, q^2, q^3) \neq (p^1, p^2, p^3)$$

AND

$$(q^1, q^2, q^3) \neq -(p^1, p^2, p^3)$$

SINCE S^2 IS HAUSDORFF WE CAN CHOOSE OPEN SETS V_p AND V_q WITH $p \in V_p$, $q \in V_q$ AND $V_p \cap V_q = \phi$. THEN $(-V_p) \cap (-V_q) = \phi$ AS WELL. BY CHOOSING THEM SMALL ENOUGH WE CAN ASSUME $V_p \cap (-V_q) = \phi$ AND $V_q \cap (-V_p) = \phi$ ALSO.



THUS, BY EXERCISE 22, $\pi(V_p)$ AND $\pi(V_q)$ ARE OPEN SETS IN $\mathbb{RP}^2$ CONTAINING $[p]$ AND $[q]$, RESPECTIVELY, AND THEY MUST BE DISJOINT BECAUSE $\pi^{-1}(\pi(V_p)) \cap \pi^{-1}(\pi(V_q)) = (V_p \cup (-V_p)) \cap (V_q \cup (-V_q))$.

NEXT WE'LL SHOW THAT $\mathbb{RP}^2$ IS <u>LOCALLY</u> <u>EUCLIDEAN</u>. SPECIFICALLY, WE WILL FIND THREE CHARTS $\{(U_k, \varphi_k)\}_{k=1,2,3}$ ON $\mathbb{RP}^2$ WITH $\bigcup_{k=1}^{3} U_k = \mathbb{RP}^2$.

FOR $k = 1, 2, 3$, LET

$$U_k = \{ [p] = [p^1, p^2, p^3] \in \mathbb{RP}^2 : p^k \neq 0 \}$$

<u>NOTE</u>: U_k IS WELL-DEFINED SINCE $p^k \neq 0$ IFF $-p^k \neq 0$.

THEN

$$\pi^{-1}(U_k) = \{ (p^1, p^2, p^3) \in S^2 : p^k \neq 0 \}$$

AND THIS IS OPEN IN S^2 SO U_k IS OPEN IN $\mathbb{RP}^2$. MOREOVER, EVERY $[p] \in \mathbb{RP}^2$ IS IN AT LEAST ONE U_k (OTHERWISE, $p^1 = p^2 = p^3 = 0$ AND $(0,0,0)$ IS NOT IN S^2) SO $\bigcup_{k=1}^{3} U_k = \mathbb{RP}^2$

NOW DEFINE

$$\varphi_k : U_k \longrightarrow \mathbb{R}^2, \quad k = 1, 2, 3$$

BY

$$\varphi_1([p]) = \varphi_1([p^1, p^2, p^3]) = \left(\frac{p^2}{p^1}, \frac{p^3}{p^1} \right)$$

$$\varphi_2([p]) = \varphi_2([p^1, p^2, p^3]) = \left(\frac{p^1}{p^2}, \frac{p^3}{p^2} \right)$$

$$\varphi_3([p]) = \varphi_3([p^1, p^2, p^3]) = \left(\frac{p^1}{p^3}, \frac{p^2}{p^3} \right)$$

<u>QUESTION</u>: WHY ARE THESE WELL-DEFINED?

WE CLAIM THAT EACH φ_k IS A HOMEOMORPHISM OF U_k ONTO $\mathbb{R}^2$. SINCE THEY ARE ALL THE SAME, I WILL PROVE THIS ONLY FOR

$$\varphi_1 : U_1 \longrightarrow \mathbb{R}^2$$

$$\varphi_1([p^1, p^2, p^3]) = \left(\frac{p^2}{p^1}, \frac{p^3}{p^1}\right)$$

φ_1 IS <u>ONE-TO-ONE</u>: SUPPOSE $[p], [q] \in U_1$ AND

$$\varphi_1([p]) = \varphi_1([q])$$

$$\left(\frac{p^2}{p^1}, \frac{p^3}{p^1}\right) = \left(\frac{q^2}{q^1}, \frac{q^3}{q^1}\right)$$

THEN

$$\frac{q^2}{q^1} = \frac{p^2}{p^1} \qquad \text{AND} \qquad \frac{q^3}{q^1} = \frac{p^3}{p^1}$$

$$q^2 = p^2\left(\frac{q^1}{p^1}\right) \qquad\qquad q^3 = p^3\left(\frac{q^1}{p^1}\right)$$

SINCE

$$q^1 = p^1\left(\frac{q^1}{p^1}\right)$$

IS OBVIOUS, WE HAVE

$$(q^1, q^2, q^3) = \left(\frac{q^1}{p^1}\right)(p^1, p^2, p^3)$$

BUT $\|q\| = \|p\| = 1$ SO $\left|\frac{q^1}{p^1}\right| = 1$, I.E., $\frac{q^1}{p^1} = \pm 1$ AND THEREFORE

$$(q^1, q^2, q^3) = \pm (p^1, p^2, p^3)$$

$$[q] = [p]$$

AS REQUIRED.

φ_1 IS <u>ONTO</u> : WE SHOW THAT EVERY (x^1, x^2) IN $\mathbb{R}^2$ CAN BE WRITTEN AS

$$(x^1, x^2) = \left(\frac{p^2}{p^1}, \frac{p^3}{p^1} \right)$$

FOR SOME $(p^1, p^2, p^3) \in S^2$ WITH $p^1 \neq 0$. BUT THIS IS EASY SINCE

$$(p^1, p^2, p^3) = \left(\frac{1}{\sqrt{1+(x^1)^2+(x^2)^2}}, \frac{x^1}{\sqrt{1+(x^1)^2+(x^2)^2}}, \frac{x^2}{\sqrt{1+(x^1)^2+(x^2)^2}} \right)$$

WILL DO THE JOB.

φ_1 IS <u>CONTINUOUS</u> : TO SHOW THAT $U_1 \subseteq \mathbb{RP}^2 \xrightarrow{\varphi_1} \mathbb{R}^2$ IS CONTINUOUS IT WILL SUFFICE, BY THE LEMMA ON PAGE 2, TO SHOW THAT

$$\varphi_1 \circ \pi : \pi^{-1}(U_1) \to \mathbb{R}^2$$

IS CONTINUOUS. BUT THIS IS GIVEN BY

$$(\varphi_1 \circ \pi)(p^1, p^2, p^3) = \left(\frac{p^2}{p^1}, \frac{p^3}{p^1} \right)$$

AND THIS IS THE RESTRICTION TO $\pi^{-1}(U_1) \subseteq S^2$ OF A MAP ON AN OPEN SET IN $\mathbb{R}^3$ $(p^1 \neq 0)$ WITH CONTINUOUS COORDINATE FUNCTIONS $(p^1, p^2, p^3) \to \frac{p^2}{p^1}$ AND $(p^1, p^2, p^3) \to \frac{p^3}{p^1}$

SO IT IS CONTINUOUS.

φ_1^{-1} IS CONTINUOUS: WE HAVE ALREADY SEEN WHAT THE MAP $\varphi_1^{-1} : \mathbb{R}^2 \to U_1$ MUST BE (PAGE 13):

$$\varphi_1^{-1}(x^1,x^2) = \left[\frac{1}{\sqrt{1+(x^1)^2+(x^2)^2}} , \frac{x^1}{\sqrt{1+(x^1)^2+(x^2)^2}} , \frac{x^2}{\sqrt{1+(x^1)^2+(x^2)^2}} \right]$$

THIS CAN BE VIEWED AS A COMPOSITION

$$\mathbb{R}^2 \longrightarrow \pi^{-1}(U_1) \xrightarrow{\pi} U_1$$

$$(x^1,x^2) \longrightarrow \left(\frac{1}{\sqrt{1+(x^1)^2+(x^2)^2}} , \frac{x^1}{\sqrt{1+(x^1)^2+(x^2)^2}} , \frac{x^2}{\sqrt{1+(x^1)^2+(x^2)^2}} \right)$$

$$\xrightarrow{\pi} \left[\frac{1}{\sqrt{1+(x^1)^2+(x^2)^2}} , \frac{x^1}{\sqrt{1+(x^1)^2+(x^2)^2}} , \frac{x^2}{\sqrt{1+(x^1)^2+(x^2)^2}} \right]$$

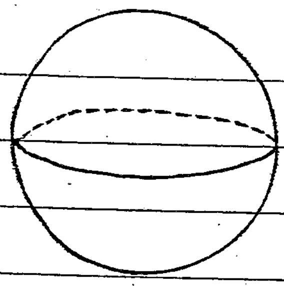
AND, SINCE BOTH OF THESE MAPS ARE CONTINUOUS, SO IS φ_1^{-1}.

<u>CONCLUSION</u>: EACH (U_k, φ_k), $k = 1,2,3$, IS A CHART ON $\mathbb{RP}^2$, WHICH IS THEREFORE LOCALLY EUCLIDEAN.

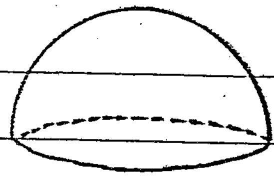
<u>EXERCISE</u> 23: COMPUTE THE OVERLAP MAPS FOR THESE CHARTS, E.G., SHOW THAT $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \{(x^1,x^2) \in \mathbb{R}^2 : x^1 \neq 0\}$ AND

$$(\varphi_2 \circ \varphi_1^{-1})(x^1,x^2) = \left(\frac{1}{x^1} , \frac{x^2}{x^1} \right).$$

CONSTRUCTING A "PICTURE" OF $\mathbb{RP}^2$:

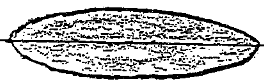


IDENTIFY OPEN LOWER HEMI- SPHERE WITH ANTIPODAL POINTS ON UPPER HEMISPHERE



ALL THAT REMAINS IS TO IDENTIFY ANTIPODAL POINTS ON THE EQUATOR

HOMEOMORPHISM



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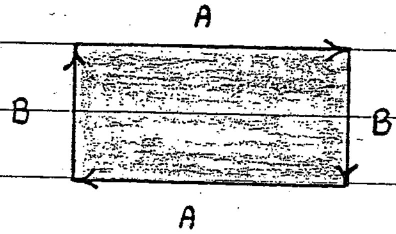
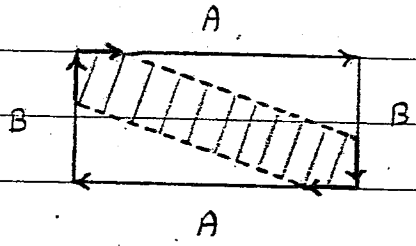
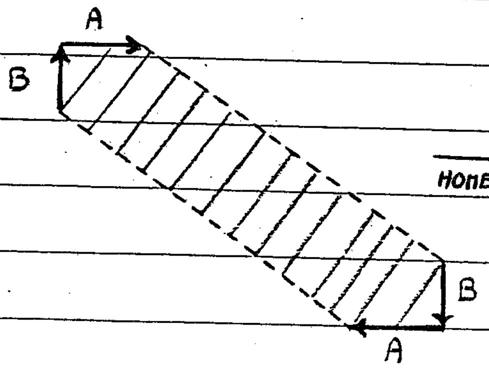


POINTS ON SIDES WITH THE SAME LABEL ARE TO BE IDENTIFIED IN SUCH A WAY THAT THE HEADS OF THE ARROWS COINCIDE

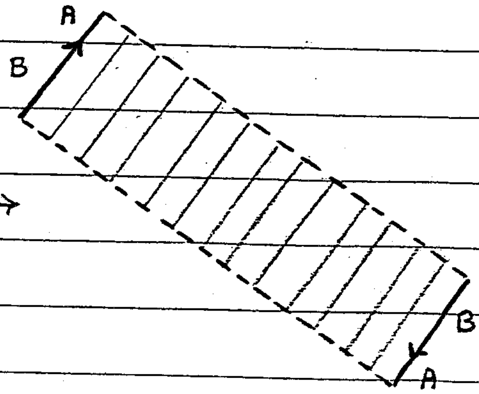
(NOW DROP THE SHADING)



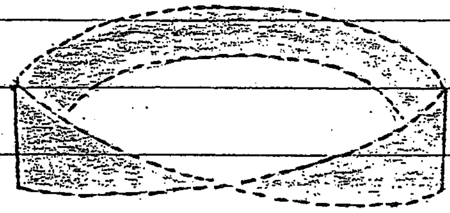
LOOK JUST AT THE DIAGONAL STRIP



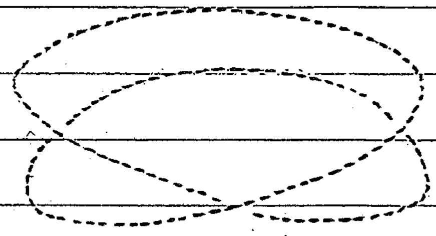
HOMEOMORPHISM



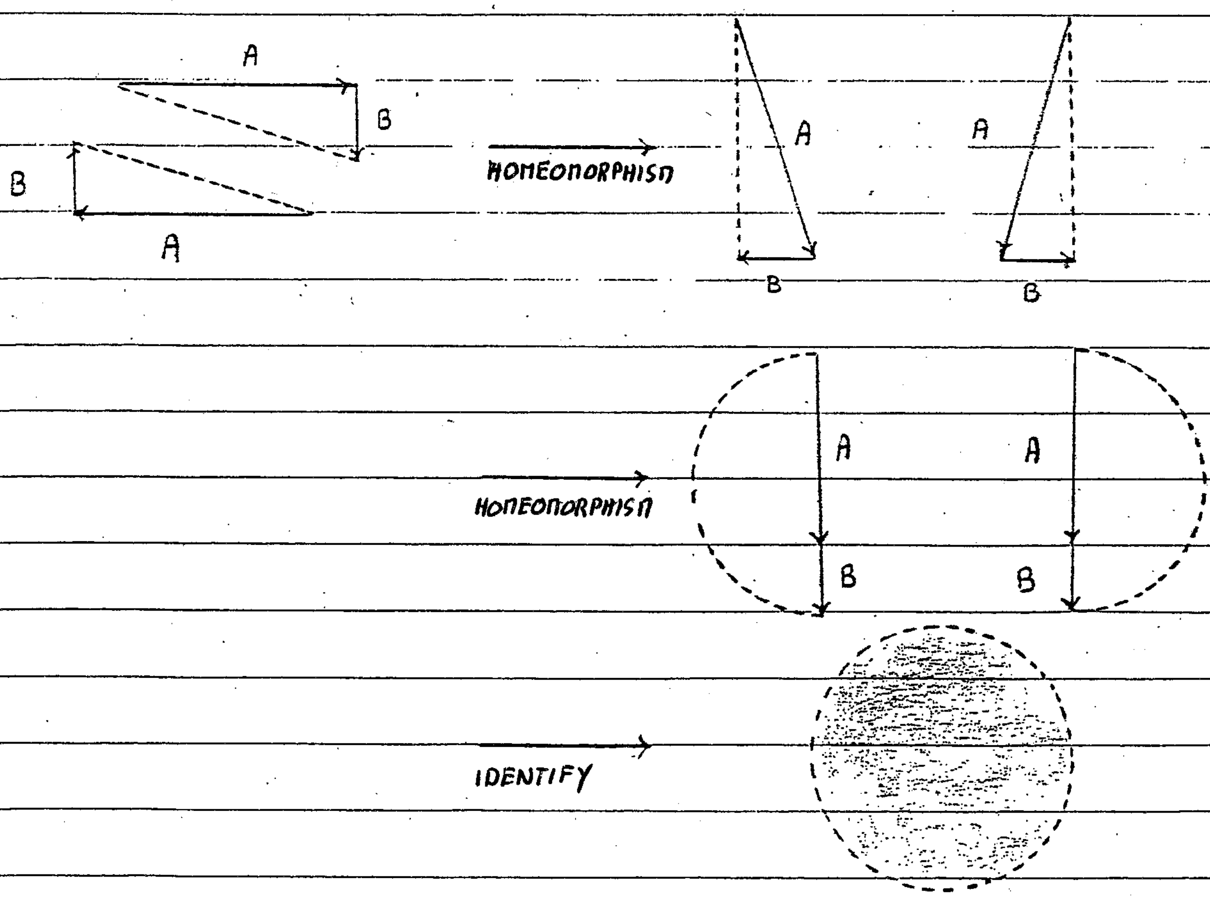
IDENTIFYING BA WITH BA AS INDICATED GIVES A MÖBIUS STRIP



NOTE : THE BOUNDARY IS A SINGLE COPY OF THE CIRCLE S^1.



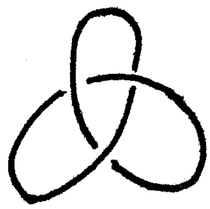
NOW EXAMINE WHAT HAPPENS TO THE REST OF THE RECTANGLE WHEN THE IDENTIFICATIONS TAKE PLACE.



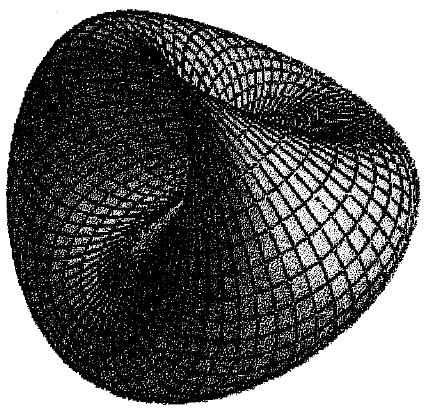
THE DOTTED CIRCLE ON THE BOUNDARY OF THE MÖBIUS STRIP AND THE DOTTED CIRCUMFERENCE OF THE DISC ARE THE SAME CIRCLE IN $\mathbb{RP}^2$ SO WE GET A "PICTURE" (SORT OF) BY SEWING THE TWO BACK TOGETHER.

> $\mathbb{RP}^2$ IS THE MÖBIUS STRIP WITH A DISC "SEWN" TO IT ALONG THEIR COMMON BOUNDARY CIRCLE.

TRY THIS SOMETIME ! YOU WILL SOON DISCOVER (AND IT CAN BE PROVED) THAT ONE CANNOT REPRESENT $\mathbb{RP}^2$ ACCURATELY (I.E., WITHOUT SELF-INTERSECTIONS) IN $\mathbb{R}^3$ IN THE SAME SENSE THAT THE KNOT



CANNOT BE REPRESENTED IN $\mathbb{R}^2$ WITHOUT SELF-INTERSECTIONS. THE BEST WE CAN DO IS SOMETHING CALLED A <u>CROSS CAP</u> :



2. (<u>REAL PROJECTIVE SPACE</u> $\mathbb{RP}^n$) IDENTIFYING ANTIPODAL POINTS ON $S^n \subseteq \mathbb{R}^{n+1}$ YIELDS, IN THE SAME WAY, THE n-DIMENSIONAL REAL PROJECTIVE SPACE $\mathbb{RP}^n$. IT IS HAUSDORFF AND LOCALLY HOMEOMORPHIC TO $\mathbb{R}^n$. THERE ARE NO PICTURES WHEN $n > 2$, BUT ONE CAN SHOW THAT $\mathbb{RP}^3$ IS HOMEOMORPHIC TO THE ROTATION GROUP $SO(3)$ OF $\mathbb{R}^3$:

$$\mathbb{RP}^3 \cong SO(3)$$

<u>EXERCISE</u> 24 : WITHOUT ATTEMPTING TO PROVE IT, OFFER A REASONABLE PLAUSIBILITY ARGUMENT FOR THIS HOMEOMORPHISM.

<u>HINT</u> : EVERY ROTATION $R \in SO(3)$ CAN BE REPRESENTED BY A VECTOR $\vec{R}$ IN $\mathbb{R}^3$ OF LENGTH $\|\vec{R}\| \leq \pi$ (R IS THE ROTATION ABOUT THE AXIS ALONG $\vec{R}$, THROUGH ANGLE $\|\vec{R}\|$ IN THE SENSE DETERMINED BY THE RIGHT-HAND RULE FROM THE DIRECTION OF $\vec{R}$). KEEP IN MIND THAT $\mathbb{RP}^3$ CAN BE THOUGHT OF AS THE RESULT OF IDENTIFYING ANTIPODAL POINTS ON THE BOUNDARY OF A SOLID 3-DIMENSIONAL BALL IN $\mathbb{R}^3$ (SEE PAGE 15).

OUR LAST EXAMPLE WILL EXHIBIT S^2 AS A CERTAIN QUOTIENT OF S^3 BY AN EQUIVALENCE RELATION WHOSE EQUIVALENCE CLASSES ARE COPIES OF S^1. THE RESULT IS ONE OF THE

MOST IMPORTANT STRUCTURES IN TOPOLOGY AND GEOMETRY.

3. (THE HOPF FIBRATION) BEGIN WITH S^3, BUT THOUGHT OF IN THE FOLLOWING WAY :

$$S^3 = \{ \xi = (z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1 \}$$

DEFINE AN EQUIVALENCE RELATION $\sim$ ON S^3 AS FOLLOWS :

$(w^1, w^2) \sim (z^1, z^2)$ IFF $\exists\, a \in \mathbb{C}$ WITH $|a| = 1$ SUCH THAT $(w^1, w^2) = (z^1 a, z^2 a)$

NOTE : IF a WERE IN $\mathbb{R}$ RATHER THAN $\mathbb{C}$, THEN $|a| = 1$ WOULD MEAN $a = \pm 1$ SO THIS WOULD BE THE EQUIVALENCE RELATION THAT IDENTIFIES ANTIPODAL POINTS OF S^3 TO GIVE $\mathbb{RP}^3$. WE ARE CONSIDERING A COMPLEX ANALOGUE OF THIS REAL PROJECTIVE SPACE.

$$\pi : S^3 \to S^3/\sim$$

$$\begin{aligned}\pi(\xi) &= [\xi] = [z^1, z^2] \\ &= \{ (z^1 a, z^2 a) : a \in \mathbb{C},\ |a| = 1 \}\end{aligned}$$

WITH THE QUOTIENT TOPOLOGY DETERMINED BY π, $S^3/\sim$ IS CALLED THE COMPLEX PROJECTIVE LINE AND DENOTED

$$\mathbb{CP}^1 .$$

HERE WE ARE IDENTIFYING TO A POINT NOT JUST A PAIR OF ANTIPODAL POINTS IN S^3, BUT THE WHOLE SET OF $(z^1 a, z^2 a)$, $a \in S^1$.

<u>CLAIM</u> : THE EQUIVALENCE CLASS OF ANY $\xi_0 = (z_0^1, z_0^2)$ IN S^3 IS A SUBSPACE

$$[\xi_0] = [z_0^1, z_0^2] = \{ (z_0^1 a, z_0^2 a) : a \in S^1 \}$$

OF S^3 HOMEOMORPHIC TO S^1.

<u>PROOF</u> : SINCE $\xi_0 \in S^3$, AT LEAST ONE OF z_0^1 OR z_0^2 MUST BE NONZERO. ASSUME, WITHOUT LOSS OF GENERALITY, THAT $z_0^1 \neq 0$.

DEFINE A MAP FROM $\mathbb{C}^2$ TO $\mathbb{C}$ BY

$$(z^1, z^2) \longrightarrow (z_0^1)^{-1} z^1 = \frac{z^1}{z_0^1}$$

IN TERMS OF REAL COORDINATES THIS IS

$$(x^1, y^1, x^2, y^2) \longrightarrow \left(\frac{x_0^1 x^1 + y_0^1 y^1}{(x_0^1)^2 + (y_0^1)^2} , \frac{x_0^1 y^1 - y_0^1 x^1}{(x_0^1)^2 + (y_0^1)^2} \right)$$

AND THIS IS CONTINUOUS.

THUS, THE RESTRICTION OF THIS MAP TO THE SUBSPACE $[z_0^1, z_0^2]$, WHICH CARRIES

$$(z_0^1 a, z_0^2 a) \longrightarrow a$$

IS ALSO CONTINUOUS. THIS RESTRICTION

$$h : [z_0^1, z_0^2] \rightarrow S^1$$

IS CLEARLY ALSO ONE-TO-ONE AND ONTO S^1. MOREOVER, THE INVERSE

$$h^{-1} : S^1 \longrightarrow [z_0^1, z_0^2]$$

IS GIVEN BY

$$h^{-1}(a) = (z_0^1 a, z_0^2 a).$$

IN TERMS OF REAL COORDINATES $(a = (s,t) \in S^1 \subseteq \mathbb{R}^2)$ THIS IS

$$h^{-1}(s,t) = (x_0^1 s - y_0^1 t,\ x_0^1 t + y_0^1 s,\ x_0^2 s - y_0^2 t,\ x_0^2 t + y_0^2 s)$$

AND THIS IS THE RESTRICTION TO S^1 OF A CONTINUOUS MAP FROM $\mathbb{R}^2$ TO $\mathbb{R}^4$ SO IT IS CONTINUOUS. THUS, h IS A HOMEOMORPHISM OF $[z_0^1, z_0^2]$ ONTO S^1. $\square$

<u>CONCLUSION</u>: OUR EQUIVALENCE RELATION CARVES S^3 UP INTO A DISJOINT UNION OF CIRCLES AND $\mathbb{CP}^1$ IS WHAT YOU GET WHEN YOU IDENTIFY EACH CIRCLE TO A POINT.

NOW, IT MIGHT SEEM THAT IDENTIFYING ALL OF THESE CIRCLES TO POINTS WOULD GIVE A RATHER COMPLICATED QUOTIENT SPACE. HOWEVER, WE WILL CONCLUDE BY SHOWING THAT, IN FACT, IT IS QUITE SIMPLE :

$$\mathbb{C}P^1 \cong S^2$$

<u>NOTE</u>: THIS REALLY ISN'T SO HARD TO BELIEVE SINCE EACH EQUIVALENCE CLASS $[z_0^1, z_0^2] = \{(z_0^1 a, z_0^2 a) : a \in S^1\}$ GIVES RISE TO A UNIQUE RATIO

$$\frac{z_0^1 a}{z_0^2 a} = \frac{z_0^1}{z_0^2}$$

WHICH CAN BE THOUGHT OF AS AN "EXTENDED COMPLEX NUMBER", I.E., ELEMENT OF THE RIEMANN SPHERE S^2.

FOR THE PROOF WE NOTICE THAT $\mathbb{C}P^1$ CAN BE SHOWN TO BE LOCALLY EUCLIDEAN IN EXACTLY THE SAME WAY AS $\mathbb{R}P^2$. SPECIFICALLY,

$$U_1 = \{[z^1, z^2] \in \mathbb{C}P^1 : z^1 \neq 0\}$$

$$\varphi_1 : U_1 \to \mathbb{C} \ (= \mathbb{R}^2)$$

$$\varphi_1([z^1, z^2]) = \frac{z^2}{z^1}$$

$$U_2 = \{[z^1, z^2] \in \mathbb{C}P^1 : z^2 \neq 0\}$$

$$\varphi_2 : U_2 \to \mathbb{C} \ (= \mathbb{R}^2)$$

$$\varphi_2([z^1, z^2]) = \frac{z^1}{z^2}$$

ARE CHARTS WHOSE DOMAINS COVER ALL OF $\mathbb{C}P^1$.

NOTE THAT U_1 COVERS ALL OF $\mathbb{CP}^1$ EXCEPT FOR THE ONE POINT

$$[0, 1]$$

(ANY $[0, z^2]$ IN $\mathbb{CP}^2$ MUST HAVE $|z^2| = 1$ SO $z^2 = 1a$ FOR SOME $a \in S^1$). φ_1 MAPS U_1 ONTO $\mathbb{R}^2$.

NOW RECALL THAT

$$\varphi_S^{-1} : \mathbb{R}^2 \longrightarrow U_S = S^2 - \{N\}$$

$$\varphi_S^{-1}(y^1, y^2) = \frac{1}{1+\|y\|^2}(2y^1, 2y^2, \|y\|^2 - 1)$$

SO

$$\varphi_S^{-1} \circ \varphi_1 : U_1 \longrightarrow S^2 - \{N\}$$

IS A HOMEOMORPHISM OF U_1 ONTO $S^2 - \{N\}$.

LET'S COMPUTE THE COMPOSITION $\varphi_S^{-1} \circ \varphi_1$: FOR $[z^1, z^2] \neq [0, 1]$,

$$\begin{aligned}
(\varphi_S^{-1} \circ \varphi_1)([z^1, z^2]) &= \varphi_S^{-1}(\varphi_1([z^1, z^2])) = \varphi_S^{-1}\left(\frac{z^2}{z^1}\right) \\
&= \varphi_S^{-1}\left(\frac{\bar{z}^1 z^2}{|z^1|^2}\right) \\
&= \frac{1}{1+\frac{|z^2|^2}{|z^1|^2}}\left(\frac{2\operatorname{Re}(\bar{z}^1 z^2)}{|z^1|^2}, \frac{2\operatorname{Im}(\bar{z}^1 z^2)}{|z^1|^2}, \frac{|z^2|^2}{|z^1|^2} - 1\right) \\
&= \frac{1}{|z^1|^2 + |z^2|^2}(2\operatorname{Re}(\bar{z}^1 z^2), 2\operatorname{Im}(\bar{z}^1 z^2), |z^2|^2 - |z^1|^2) \\
&= (2\operatorname{Re}(\bar{z}^1 z^2), 2\operatorname{Im}(\bar{z}^1 z^2), |z^2|^2 - |z^1|^2)
\end{aligned}$$

THUS, THE HOMEOMORPHISM $\varphi_S^{-1} \circ \varphi_1 : U_1 \to S^2 - \{N\}$ IS GIVEN BY

$$(\varphi_S^{-1} \circ \varphi_1)([z^1, z^2]) = \left(2\,\mathrm{Re}(\bar{z}^1 z^2),\ 2\,\mathrm{Im}(\bar{z}^1 z^2),\ |z^2|^2 - |z^1|^2\right)$$

FOR $[z^1, z^2] \neq [0,1]$.

BUT NOTICE THAT THE MAP

$$[z^1, z^2] \longrightarrow \left(2\,\mathrm{Re}(\bar{z}^1 z^2),\ 2\,\mathrm{Im}(\bar{z}^1 z^2),\ |z^2|^2 - |z^1|^2\right)$$

IS ACTUALLY DEFINED AT $[0,1]$ AND, IN FACT,

$$[0,1] \to (0,0,1) = N$$

SO THIS MAP CARRIES $\mathbb{CP}^1$ ONTO S^2. IT IS CONTINUOUS BECAUSE ITS COMPOSITION WITH $(z^1, z^2) \xrightarrow{\pi} [z^1, z^2]$ IS CONTINUOUS (WRITE IT OUT IN REAL COORDINATES). IT IS ONE-TO-ONE BECAUSE $\varphi_S^{-1} \circ \varphi_1$ IS ONE-TO-ONE AND N IS THE IMAGE ONLY OF $[0,1]$. ONE CAN EITHER VERIFY DIRECTLY THAT ITS INVERSE IS ALSO CONTINUOUS, OR ONE CAN BE PATIENT FOR A LITTLE WHILE UNTIL WE SEE THAT THIS FOLLOWS DIRECTLY FROM AN IMPORTANT PROPERTY OF $\mathbb{CP}^1$ ("COMPACTNESS") THAT WE WILL DISCUSS SOON. IN EITHER CASE WE NOW HAVE

$$\mathbb{CP}^1 \cong S^2 .$$

COMPOSING THE QUOTIENT MAP $S^3 \longrightarrow \mathbb{CP}^1$ WITH THIS HOMEOMORPHISM $\mathbb{CP}^1 \to S^2$ WE HAVE

$$\begin{array}{c} S^3 \\ \Big\downarrow H \\ S^2 \end{array}$$

$$H(z^1, z^2) = (2\,\mathrm{Re}(\bar{z}^1 z^2),\ 2\,\mathrm{Im}(\bar{z}^1 z^2),\ |z^2|^2 - |z^1|^2)$$

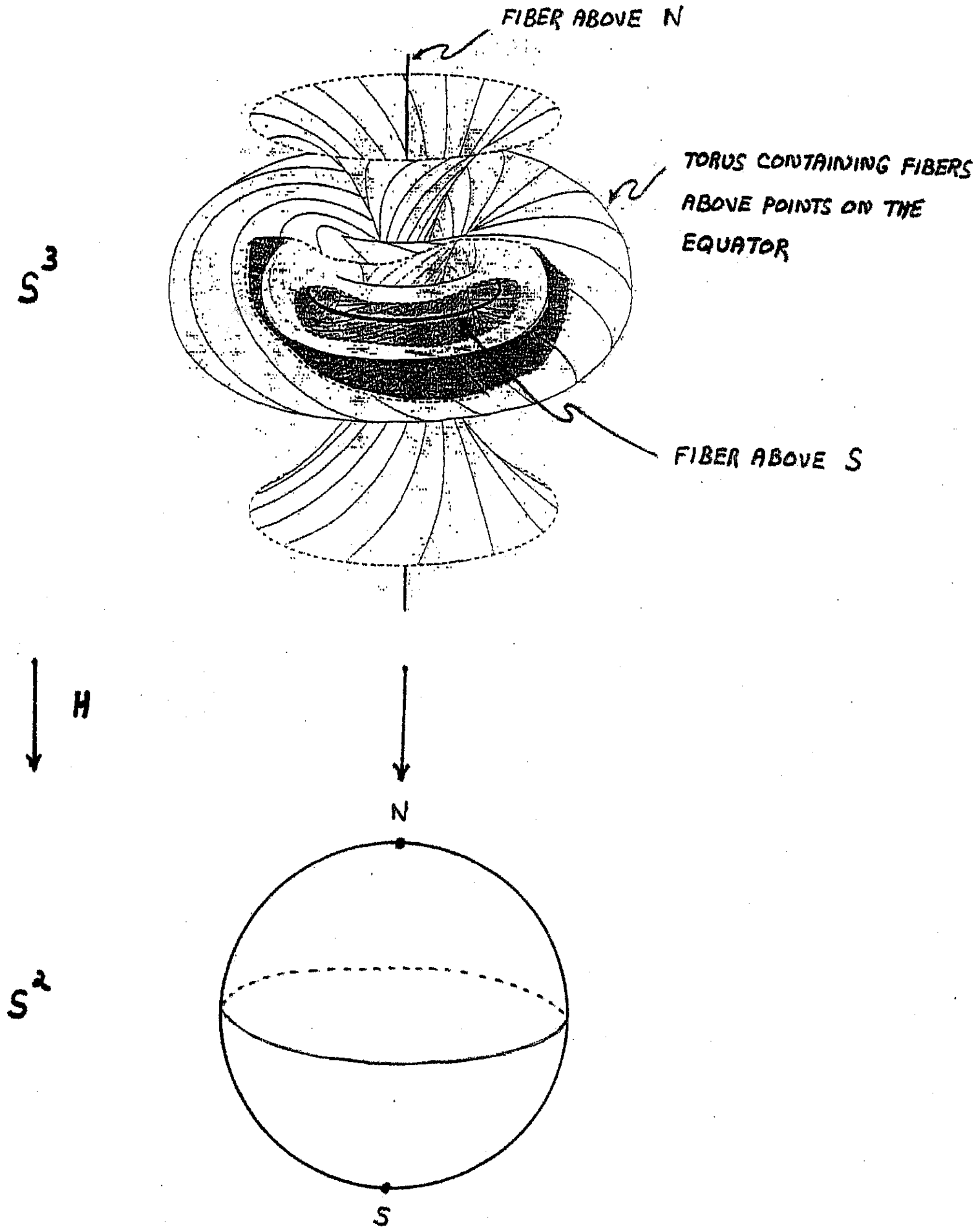
THIS IS CALLED THE HOPF MAP. FOR EACH $p \in S^2$, THE FIBER $H^{-1}(p)$ ABOVE p IS A SUBSPACE OF S^3 HOMEOMORPHIC TO S^1.

WE HAVE CONSTRUCTED WHAT IS CALLED THE HOPF FIBRATION OF S^3 BY S^1 OVER S^2. THIS OBJECT IS OF CRUCIAL IMPORTANCE IN TOPOLOGY (HOMOTOPY GROUPS), GEOMETRY (CONNECTIONS ON PRINCIPAL BUNDLES) AND, ODDLY ENOUGH, PHYSICS (GAUGE THEORY).

WITH A LITTLE EFFORT IT IS POSSIBLE TO JUSTIFY THE FOLLOWING "PICTURE" OF THE HOPF FIBRATION (ALSO CALLED THE COMPLEX HOPF BUNDLE) :

EXERCISE 25 : DESCRIBE HOW YOU MIGHT "PICTURE" THE 3-SPHERE S^3. HINT: DON'T TRY TO DO IT IN $\mathbb{R}^4$ (YOU'LL ONLY HURT YOURSELF). RATHER, THINK ABOUT HOW THE (RIEMANN) SPHERE S^2 IS PICTURED IN COMPLEX ANALYSIS VIA STEREOGRAPHIC PROJECTION.

COMPLEX HOPF BUNDLE



WE'LL CONCLUDE THIS SECTION WITH A FEW EXERCISES ON A VERY IMPORTANT SPECIAL TYPE OF QUOTIENT MAP.

<u>MOTIVATION</u> : NOTICE (FOR EXAMPLE, FROM PAGE 10) THAT $\pi : S^2 \to \mathbb{RP}^2$ HAS THE FOLLOWING PROPERTY : EVERY $[p] \in \mathbb{RP}^2$ IS CONTAINED IN SOME OPEN SET U WITH THE PROPERTY THAT $\pi^{-1}(U)$ IS A DISJOINT UNION $U_1 \sqcup U_2$ OF OPEN SETS IN S^2 EACH OF WHICH IS MAPPED

HOMEOMORPHICALLY BY π ONTO U.

COVERING SPACES : A CONTINUOUS SURJECTION

$$p : X \to Y$$

IS CALLED A COVERING MAP IF EACH $y \in Y$ IS CONTAINED IN SOME OPEN SUBSET U OF Y SUCH THAT $p^{-1}(U)$ IS A DISJOINT UNION $\bigsqcup_{\alpha \in A} U_\alpha$ OF OPEN SETS U_α IN X SUCH THAT $p|U_\alpha : U_\alpha \to U$ IS A HOMEOMORPHISM FOR EACH $\alpha \in A$.

EXERCISE 26 : SHOW THAT EVERY COVERING MAP IS AN OPEN MAP (SO Y MUST HAVE THE QUOTIENT TOPOLOGY DETERMINED BY p).

EXERCISE 27 : SHOW THAT

$$p : \mathbb{R} \to S^1$$

$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

IS A COVERING MAP

EXERCISE 28 : LET $k \geq 1$ BE AN INTEGER. SHOW THAT

$$p : S^1 \to S^1$$

$$p(z) = z^k$$

IS A COVERING MAP.