RELATED RATES PROBLEMS

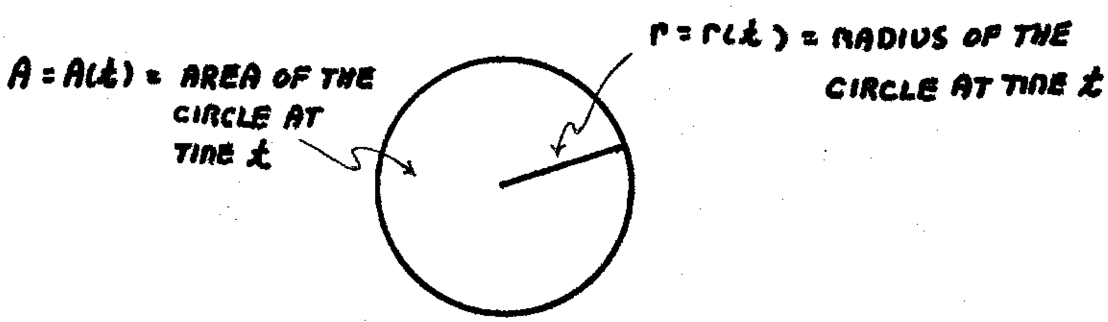
TO ILLUSTRATE THE GENERAL FEATURES OF THESE PROBLEMS WE BEGIN WITH A SIMPLE EXAMPLE.

A STONE IS DROPPED INTO A POOL OF WATER. A CIRCULAR RIPPLE SPREADS OUT IN THE POOL. THE RADIUS INCREASES AT A RATE OF 3 in/sec. HOW FAST IS THE AREA OF THE CIRCLE INCREASING ?

> <u>GENERAL FEATURES</u> : TWO RATES OF CHANGE INVOLVED. ONE KNOWN, ONE TO BE DETERMINED. NEED A RELATIONSHIP BETWEEN THESE RATES OF CHANGE SO WE CAN PLUG IN THE ONE WE KNOW AND SOLVE FOR THE ONE WE DON'T KNOW.

<u>PROCEDURE</u> (ILLUSTRATED ON THE EXAMPLE ABOVE) :

1. DRAW A "SNAPSHOT" AT SOME TYPICAL INSTANT t AND INTRODUCE NAMES FOR THE VARIABLES WHOSE RATES OF CHANGE (DERIVATIVES) ARE INVOLVED.



2. WRITE OUT EXPLICITLY WHAT YOU KNOW AND WHAT YOU'RE SUPPOSED TO FIND.

KNOWN : $\frac{dr}{dt} = 3$ in/sec

FIND : $\frac{dA}{dt}$

3. FIND A RELATIONSHIP BETWEEN THE VARIABLES THEMSELVES THAT IS TRUE AT ALL TIMES t.

$$A = \pi r^2$$

4. DIFFERENTIATE THIS RELATIONSHIP WITH RESPECT TO t.

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt}$$

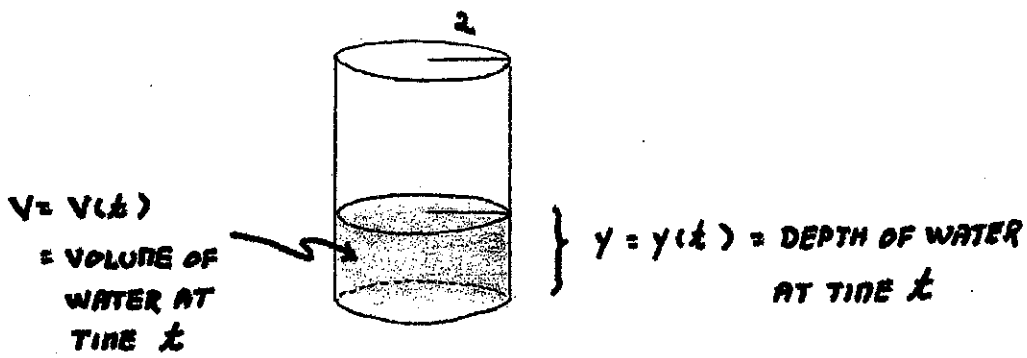
5. PLUG IN THE DERIVATIVE YOU KNOW.

$$\frac{dA}{dt} = 2\pi r(3) = 6\pi r \text{ in}^2/\text{sec}$$

<u>NOTE</u> : IN THIS EXAMPLE, EVEN THOUGH r INCREASES AT A CONSTANT RATE, A DOES NOT, I.E., $\frac{dA}{dt}$ DEPENDS ON r. THIS MAKES SENSE, RIGHT? THE LARGER THE CIRCLE THE GREATER THE EFFECT ON THE AREA OF A 3in CHANGE IN THE RADIUS.

MORE EXAMPLES :

1. HOW FAST DOES THE WATER LEVEL RISE IN A CYLINDRICAL CAN OF RADIUS 2 FT IF WATER IS BEING POURED IN AT A RATE OF 3 FT^3/SEC ?



KNOWN : $\frac{dV}{dt} = 3$

FIND : $\frac{dy}{dt}$

RELATIONSHIP BETWEEN $V(t)$ AND $y(t)$ (FORMULA FOR THE VOLUME OF A CYLINDER IS $V = \pi r^2 h$) :

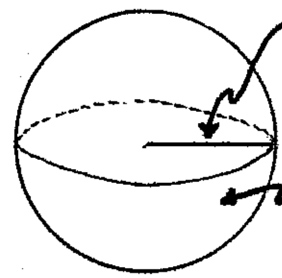
$$V = \pi (2^2) y$$
$$V = 4\pi y$$

DIFFERENTIATE WITH RESPECT TO t :

$$\frac{dV}{dt} = 4\pi \frac{dy}{dt}$$

SO $\frac{dy}{dt} = \frac{\frac{dV}{dt}}{4\pi} = \frac{3}{4\pi}$ FT/SEC

2. GAS IS ESCAPING FROM A SPHERICAL BALLOON AT A RATE OF 2 FT^3/MIN. HOW FAST IS THE RADIUS DECREASING WHEN IT (THE RADIUS) IS 12 FT?



KNOWN : $\frac{dV}{dt} = -2$ (NOTE THE MINUS SIGN)

FIND : $\frac{dr}{dt}$ WHEN $r = 12$

RELATIONSHIP BETWEEN $V(t)$ AND $r(t)$ (FORMULA FOR THE VOLUME OF A SPHERE) :

$$V = \frac{4}{3}\pi r^3$$

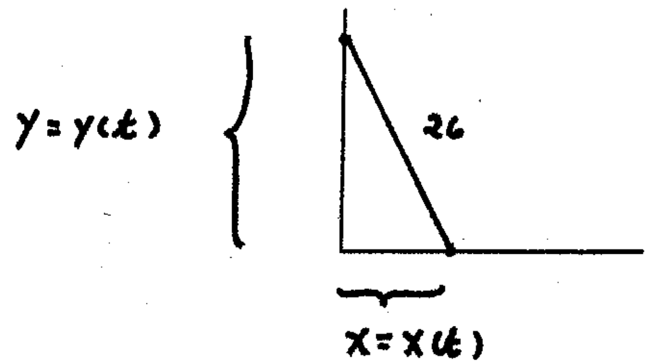
DIFFERENTIATE WITH RESPECT TO t :

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$$

SO

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} = \frac{-2}{4\pi(12^2)} = -\frac{1}{288\pi} \text{ FT/MIN}$$

3. A LADDER 26 FT LONG RESTS ON THE (HORIZONTAL) GROUND AND LEANS AGAINST A (VERTICAL) WALL. THE BASE OF THE LADDER IS PULLED AWAY FROM THE WALL AT A RATE OF 4 FT/SEC. HOW FAST IS THE TOP OF THE LADDER SLIDING DOWN THE WALL WHEN THE BASE IS 10 FT FROM THE WALL?



KNOWN : $\frac{dx}{dt} = 4$

FIND : $\frac{dy}{dt}$ WHEN $x = 10$

RELATIONSHIP BETWEEN $x(t)$ AND $y(t)$ (PYTHAGOREAN THEOREM):

$$x^2 + y^2 = 26^2$$

DIFFERENTIATE WITH RESPECT TO t :

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

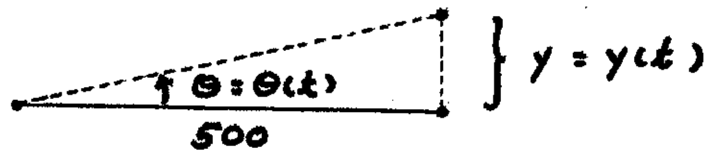
SO

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

<u>NOTE</u> : WHEN $x = 10$, $x^2 + y^2 = 26^2$ GIVES $10^2 + y^2 = 26^2$ SO $y = 24$.

$$\frac{dy}{dt} = -\frac{10}{24}(4) = -\frac{5}{3} \text{ FT/SEC}$$

4. A BALLOON IS RISING STRAIGHT UP FROM A LEVEL FIELD AND IS BEING TRACKED BY A CAMERA 500 FT FROM THE POINT OF LIFT OFF. AT THE INSTANT WHEN THE CAMERA'S ANGLE OF ELEVATION IS $\frac{\pi}{4}$, THAT ANGLE IS INCREASING AT A RATE OF 0.14 RADIANS/MIN. HOW FAST IS THE BALLOON RISING AT THAT INSTANT ?



KNOWN : $\frac{d\theta}{dt} = 0.14$ WHEN $\theta = \frac{\pi}{4}$

FIND : $\frac{dy}{dt}$ WHEN $\theta = \frac{\pi}{4}$

RELATIONSHIP BETWEEN $\theta(t)$ AND $y(t)$ (DEFINITION OF $\tan\theta$)

$$\tan\theta = \frac{y}{500} \Rightarrow$$

$$y = 500 \tan\theta$$

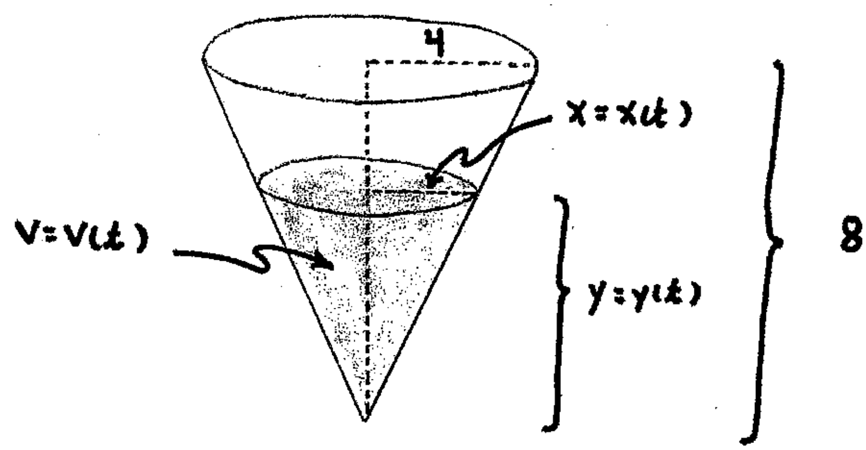
DIFFERENTIATE WITH RESPECT TO t :

$$\frac{dy}{dt} = 500 \sec^2\theta \frac{d\theta}{dt}$$

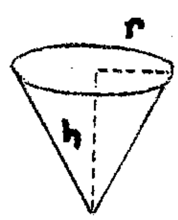
$$= 500\,(2)\,(0.14)$$

$$= 140 \text{ FT/MIN}$$

5. WATER IS RUNNING OUT OF A CONICAL FUNNEL AT A RATE OF $1 \text{ IN}^3/\text{SEC}$. IF THE ALTITUDE OF THE FUNNEL IS 8 IN AND THE RADIUS OF ITS BASE IS 4 IN, HOW FAST IS THE WATER LEVEL DROPPING WHEN IT IS 2 IN FROM THE TOP?



<u>NOTE</u>: IT MAY NOT BE OBVIOUS AT THIS POINT THAT WE NEED $x = x(t)$ (RADIUS OF THE WATER LEVEL AT TIME t) SINCE $\frac{dV}{dt}$ IS GIVEN AND $\frac{dy}{dt}$ IS REQUESTED. THE REASON IS IN THE FORMULA FOR THE VOLUME OF A CONE



$$V = \frac{1}{3}\pi r^2 h$$

WHICH REQUIRES BOTH THE ALTITUDE AND THE RADIUS.

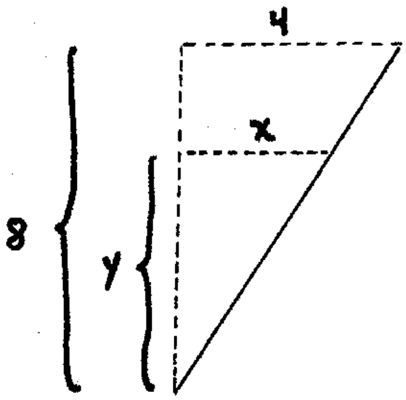
KNOWN: $\frac{dV}{dt} = -1$

FIND: $\frac{dy}{dt}$ WHEN $y = 8 - 2 = 6$

RELATIONSHIP BETWEEN $V(t)$ AND $y(t)$ (FORMULA FOR THE VOLUME OF A CONE):

$$V = \frac{1}{3}\pi x^2 y$$

SINCE WE HAVE NO INFORMATION ABOUT x OR $\frac{dx}{dt}$ WE WILL WRITE x IN TERMS OF y USING <u>SIMILAR TRIANGLES</u>:



$$\frac{x}{y} = \frac{4}{8}$$

$$\Rightarrow x = \frac{1}{2} y$$

THUS,

$$V = \frac{1}{3}\pi\left(\frac{1}{2}y\right)^2 y$$

$$V = \frac{\pi}{12} y^3$$

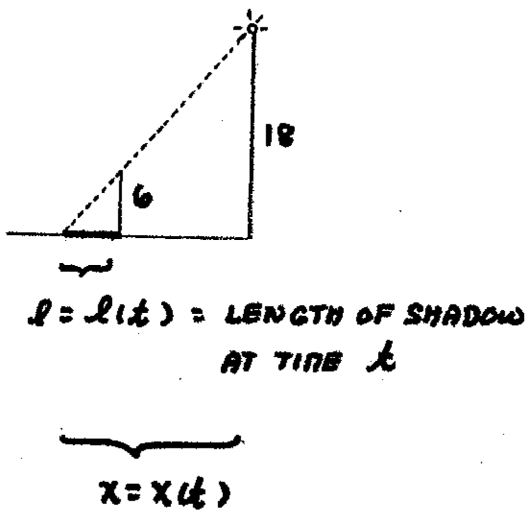
DIFFERENTIATE WITH RESPECT TO t:

$$\frac{dV}{dt} = \frac{\pi}{12}\left(3y^2\frac{dy}{dt}\right) = \frac{\pi}{4}y^2\frac{dy}{dt}$$

SO

$$\frac{dy}{dt} = \frac{\frac{dV}{dt}}{\frac{\pi}{4}y^2} = \frac{-1}{\frac{\pi}{4}(6^2)} = -\frac{1}{9\pi} \text{ in/SEC}$$

6. A MAN 6 FT TALL IS WALKING AT 3FT/SEC TOWARD A STREETLIGHT 18 FT HIGH. HOW FAST IS THE LENGTH OF HIS SHADOW CHANGING?



SIMILAR TRIANGLES $\Rightarrow \quad \frac{\ell}{6} = \frac{x}{18} \quad \Rightarrow \ell = \frac{1}{3}x$

KNOWN : $\quad \frac{dx}{dt} = -3$

FIND : $\quad \frac{d\ell}{dt}$

RELATIONSHIP BETWEEN $\ell(t)$ AND $x(t)$: $\quad \ell = \frac{1}{3}x$

DIFFERENTIATE WITH RESPECT TO t :

$$\frac{d\ell}{dt} = \frac{1}{3}\frac{dx}{dt} = \frac{1}{3}(-3) = -1 \text{ FT/SEC}$$