REVIEW OF FUNCTIONS

THE FORMULA $A = \pi r^2$ FOR THE AREA OF A CIRCLE IS A RULE WHICH ASSOCIATES WITH EACH POSITIVE REAL NUMBER r EXACTLY ONE OTHER NUMBER A, NAMELY, THE AREA OF A CIRCLE OF RADIUS r.

A REAL-VALUED FUNCTION OF A REAL VARIABLE (OR JUST FUNCTION FOR SHORT) CONSISTS OF TWO THINGS:

(1) A SET D OF REAL NUMBERS CALLED THE DOMAIN

(2) A RULE WHICH ASSOCIATES WITH EVERY REAL NUMBER x IN THE DOMAIN D EXACTLY ONE NUMBER y CALLED THE VALUE OF THE FUNCTION AT x.

THE SET OF ALL VALUES OF THE FUNCTION IS CALLED ITS RANGE.

THE RULES ARE GENERALLY GIVEN NAMES LIKE

$$f, g, \ldots, F, G, \ldots$$

IF THE NAME IS f, THEN THE VALUE OF f AT x IS OFTEN WRITTEN

$$f(x) \qquad (\text{" } f \text{ of } x \text{ "})$$

RATHER THAN y:

$$y = f(x)$$

SOME SPECIAL FUNCTIONS HAVE NAMES RESERVED FOR THEM, E.G.,

$$\sin,\ \cos,\ \ldots,\ \ln,\ \exp,\ \ldots,\ \arctan,\ \ldots$$

SOME METHODS OF DESCRIBING FUNCTIONS :

1. FORMULAS

E.G., $f(x) = \frac{1}{\sqrt{1-x^2}}$ IN THIS CASE THE DOMAIN IS GENERALLY TAKEN TO BE THE NATURAL DOMAIN, I.E., THE LARGEST SET OF REAL NUMBERS x TO WHICH THE FORMULA CAN BE APPLIED TO GIVE ANOTHER REAL NUMBER. FOR $f(x) = \frac{1}{\sqrt{1-x^2}}$ THIS WOULD BE

$$-1 < x < 1.$$

EXCEPTION : THE FORMULA MAY ARISE IN SOME CONTEXT THAT PLACES FURTHER RESTRICTIONS ON THE DOMAIN, E.G., $A = \pi r^2$ MAKES PERFECT SENSE FOR $r \leq 0$, BUT CIRCLES DO NOT HAVE SUCH RADII.

A FEW SPECIAL CLASSES OF FUNCTIONS DESCRIBED BY FORMULAS :

POWER FUNCTIONS :

$$x^5$$
$$x^{\frac{3}{2}} \qquad (= (\sqrt{x})^3)$$
$$\sqrt[3]{x} \qquad (= \sqrt[3]{x})$$
$$x^{-1.2}$$
$$x^{\pi}$$
$$\vdots$$
$$x^{a}$$

POLYNOMIALS :

$$5x + 3$$

$$x^2 - 2x + 1$$

$$6x^3 - 7$$

$$8$$

$$4x^4 - \tfrac{3}{2}x^3 + 7x^2 + 5x - 2$$

$$\vdots$$

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

RATIONAL FUNCTIONS :

$$\frac{5x+3}{x^2-2x+1}$$

$$\frac{x^2-4}{5x}$$

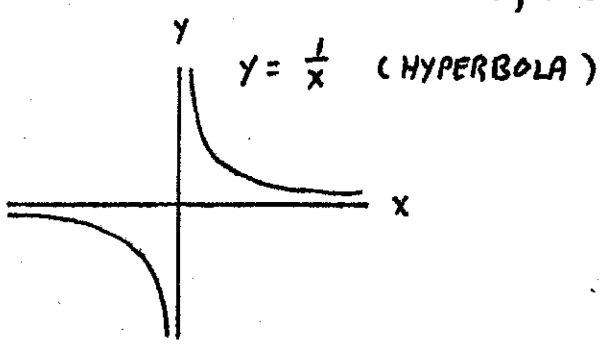
$$\frac{1}{x}$$

$$\vdots$$

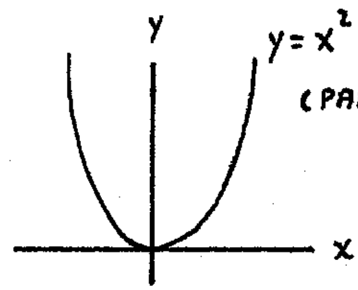
$$\frac{P(x)}{Q(x)}$$ WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS

2. GRAPHS

ALL FUNCTIONS HAVE GRAPHS, E.G.,

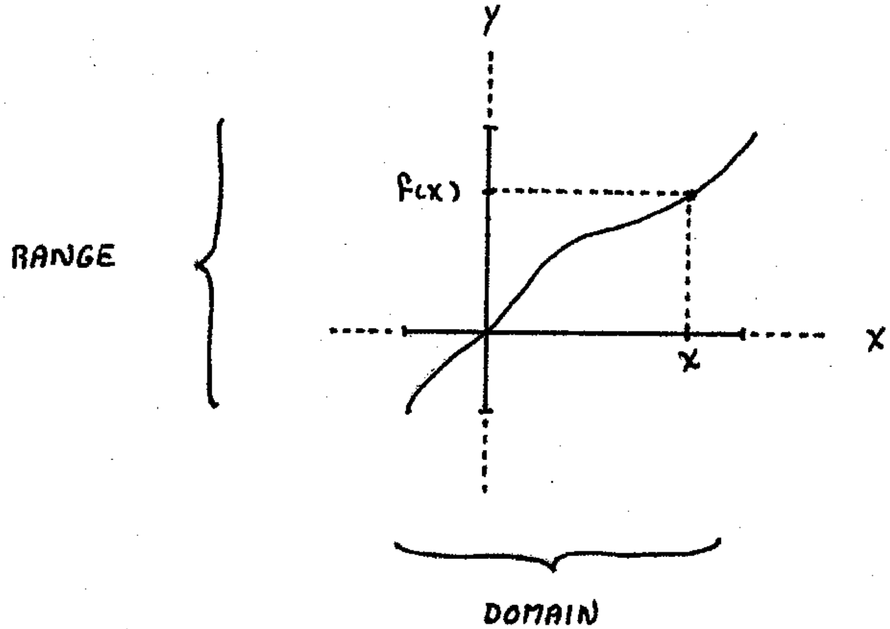


, ETC.



$y = x^2$ (PARABOLA)

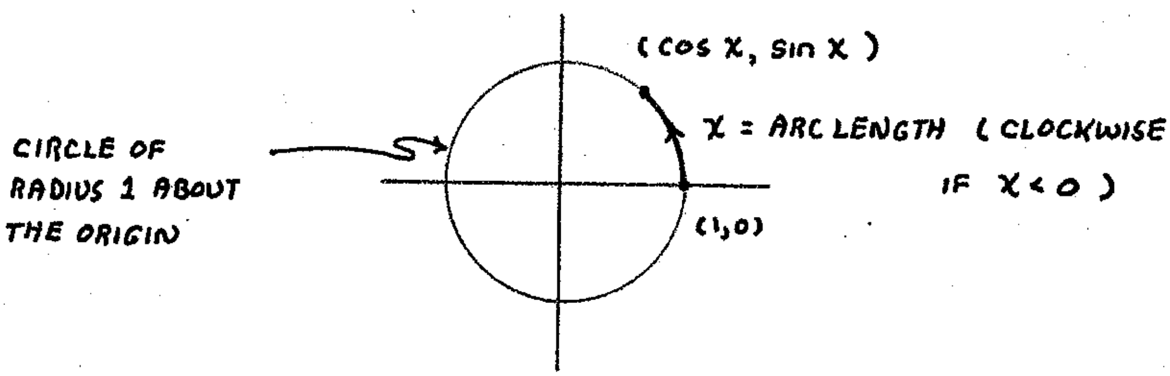
ON THE OTHER HAND, <u>ANY</u> CURVE IN THE XY-PLANE THAT PASSES THE "VERTICAL LINE TEST" DEFINES A FUNCTION WHOSE VALUE AT X IS THE HEIGHT OF THE CURVE ABOVE (OR BELOW) X.



<u>VERTICAL LINE TEST</u>: <u>NO VERTICAL LINE HITS THE CURVE MORE THAN ONCE</u>.

3. <u>GEOMETRICALLY</u>

FOR EXAMPLE, cos X AND sin X ARE DEFINED BY THE PICTURE



THEN THE REST:

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

SOME SPECIAL VALUES YOU MUST KNOW :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0

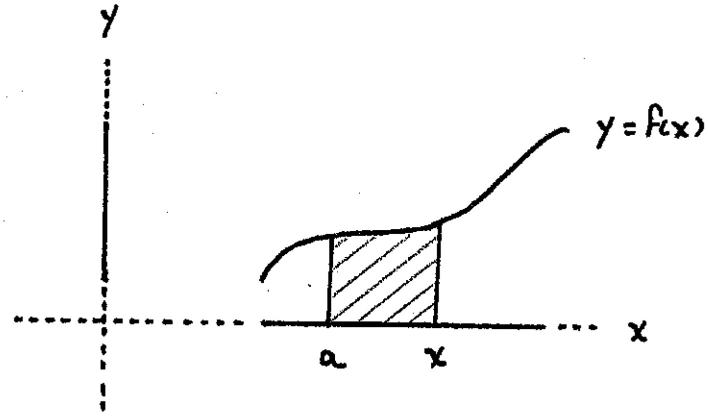
SOME IDENTITIES YOU MUST KNOW :

$$\cos^2 x + \sin^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

HERE'S ANOTHER (SEEMINGLY ODD, BUT <u>VERY</u> IMPORTANT) GEOMETRICAL WAY OF DEFINING FUNCTIONS :

SUPPOSE $f(x) \geq 0$ ON SOME INTERVAL I

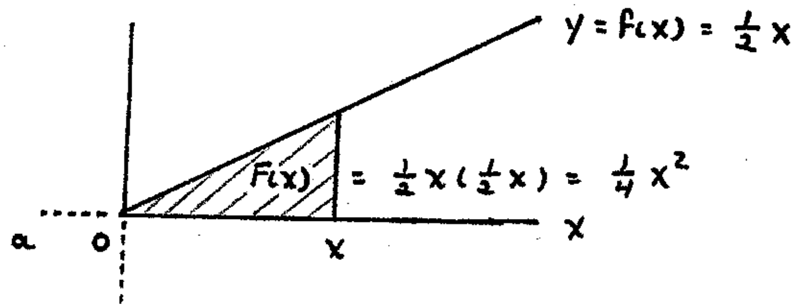
AND FIX SOME a IN I.



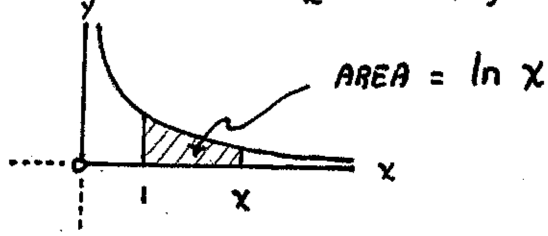
FOR ANY x IN I DEFINE

$$F(x) = \begin{cases} \text{AREA UNDER } y = f(x) \text{ FROM } a \text{ TO } x, & \text{IF } x \geq a \\ -\text{ AREA UNDER } y = f(x) \text{ FROM } x \text{ TO } a, & \text{IF } x < a \end{cases}$$

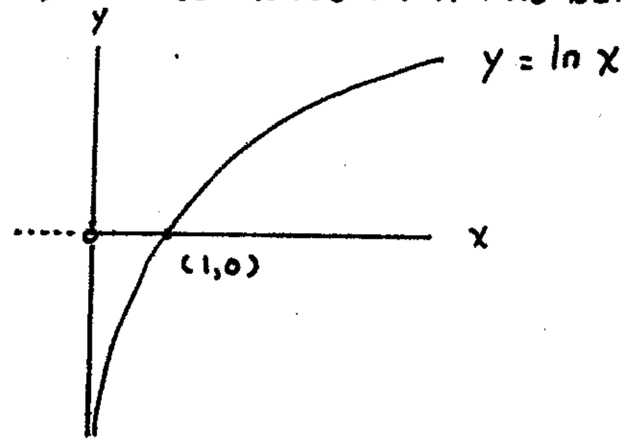
E.G., IF $y = f(x) = \frac{1}{2}x$ ON $I = [0, \infty)$ WITH $a = 0$



SPECIAL CASE : WE WILL SEE LATER THAT THE "PROPER" DEFINITION OF THE NATURAL LOGARITHM FUNCTION $\ln x$, FOR $x > 0$, IS AS THE "AREA FUNCTION" FOR $f(x) = \frac{1}{x}$ ON $(0, \infty)$ WITH $a = 1$:



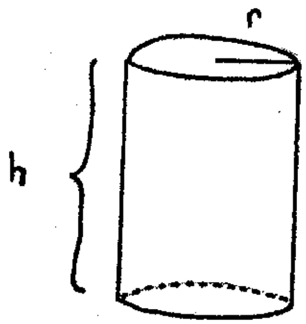
ALL OF THE "USUAL" PROPERTIES OF THE NATURAL LOGARITHM (WHICH YOU MUST KNOW) CAN BE PROVED FROM THIS DEFINITION:



$$\ln(1) = 0$$
$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\ln(a^r) = r \ln a$$

4. APPLICATIONS

EXAMPLE: A SOUP COMPANY WANTS TO MANUFACTURE A TIN CAN IN THE SHAPE OF A RIGHT CIRCULAR CYLINDER () THAT WILL HOLD 500 cm^3 OF SOUP. EXPRESS THE AMOUNT OF TIN REQUIRED TO BUILD THE CAN AS A FUNCTION OF THE RADIUS r OF THE CAN (AND THINK ABOUT THE PROBLEM OF FINDING A CHOICE OF r THAT WILL MINIMIZE THE AMOUNT OF MATERIAL REQUIRED).



SURFACE AREA $A = \pi r^2 + \pi r^2 + 2\pi r h$

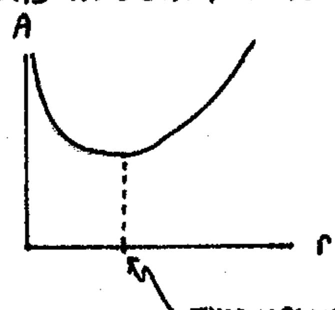
$$A = 2\pi r^2 + 2\pi r h$$

VOLUME $= 500 = \pi r^2 h \Rightarrow h = \dfrac{500}{\pi r^2}$

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{1000}{r}, \quad r > 0$$

THE GRAPH OF $A(r)$ LOOKS ROUGHLY LIKE THIS

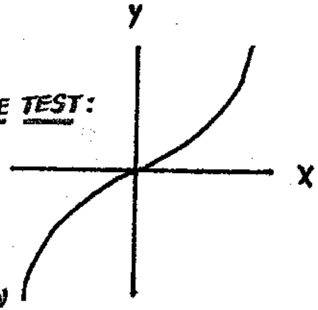


THIS VALUE OF r WOULD REQUIRE THE LEAST AMOUNT OF TIN. WE WILL FIND IT LATER.

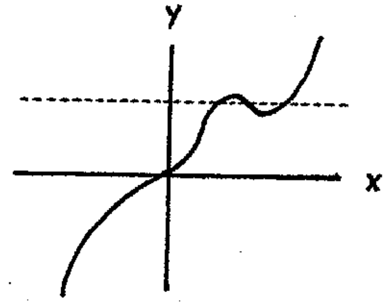
5. INVERSES

A FUNCTION f WITH DOMAIN D IS SAID TO BE ONE-TO-ONE ON D IF ITS GRAPH PASSES THE "HORIZONTAL LINE TEST".

HORIZONTAL LINE TEST: NO HORIZONTAL LINE HITS THE GRAPH MORE THAN ONCE.

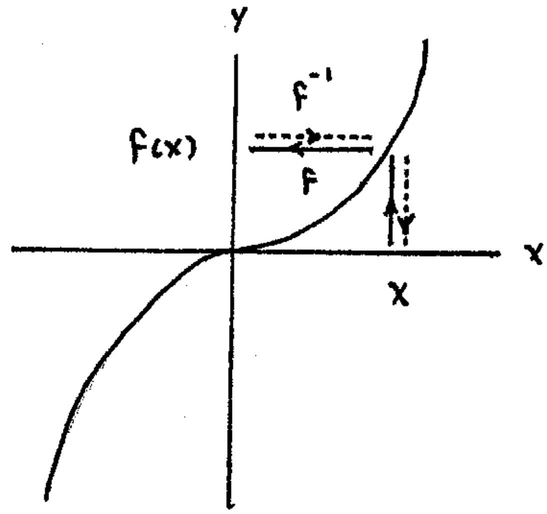


ONE - TO - ONE



NOT ONE - TO - ONE

IF f IS ONE-TO-ONE, THEN WE CAN DEFINE ANOTHER FUNCTION, WRITTEN f^{-1} AND CALLED THE <u>INVERSE</u> OF f, WHICH "UNDOES" f:

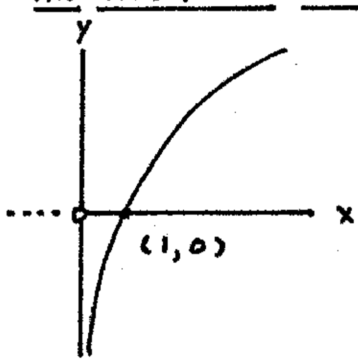


$$f^{-1}(f(x)) = x$$

$$x \xrightarrow{f} f(x) \xrightarrow{f^{-1}} x$$

E.G., IF $f(x) = x^3$ ON $(-\infty, \infty)$, THEN $f^{-1}(x) = \sqrt[3]{x}$ BECAUSE $f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x$.

<u>AN IMPORTANT EXAMPLE</u>: $f(x) = \ln x$ ON $(0, \infty)$



THERE IS NO SIMPLE FORMULA FOR THE INVERSE $\ln^{-1}$ OF $\ln$, BUT IT HAS A NAME AND A SYMBOL (ACTUALLY, TWO SYMBOLS)

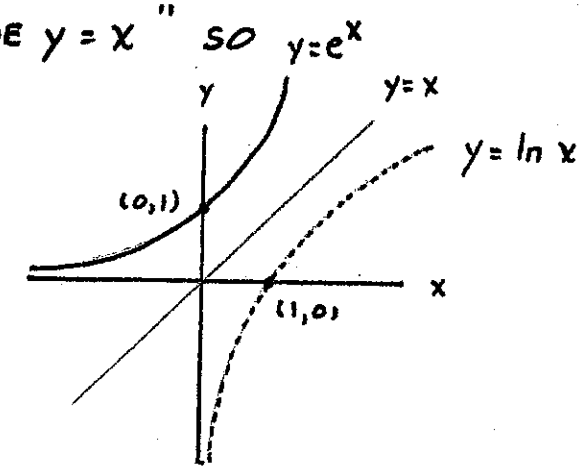
$\ln^{-1}$ IS CALLED THE <u>EXPONENTIAL</u> <u>FUNCTION</u> AND ITS VALUE AT ANY x IN $(-\infty, \infty)$ IS WRITTEN

$$\exp(x)$$

OR

$$e^x .$$

GRAPHS OF INVERSE FUNCTIONS ARE OBTAINED BY "REFLECTING ACROSS THE LINE $y = x$" SO



SINCE $\ln x$ AND e^x ARE INVERSES,

$$e^{\ln x} = x \quad , \quad 0 < x < \infty$$

$$\ln(e^x) = x \quad , \quad -\infty < x < \infty$$

THE PROPERTIES OF $\ln x$ LISTED EARLIER TRANSLATE INTO THE FAMILIAR PROPERTIES OF THE EXPONENTIAL FUNCTION (WHICH YOU MUST KNOW):

$$e^0 = 1$$

$$e^a e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

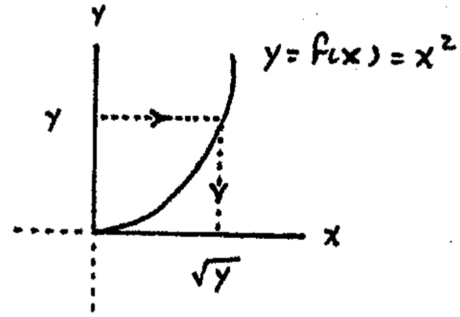
$$(e^a)^r = e^{ar}$$

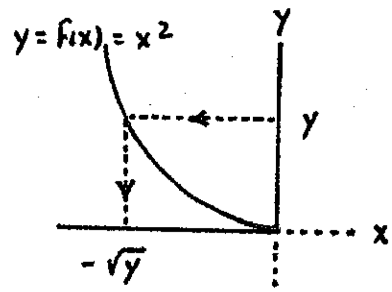
FUNCTIONS THAT ARE NOT ONE-TO-ONE ON THEIR ENTIRE DOMAINS OFTEN BECOME ONE-TO-ONE WHEN THEIR DOMAINS ARE RESTRICTED SO THEY HAVE INVERSES ON THESE RESTRICTED DOMAINS.

<u>EXAMPLES</u>:

1. $y = f(x) = x^2$ ON $[0, \infty)$

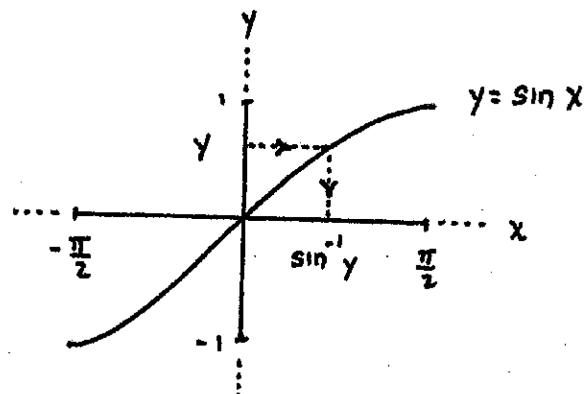
   $y = f^{-1}(x) = \sqrt{x}$ ON $[0, \infty)$





2. $y = f(x) = x^2$ ON $(-\infty, 0]$

   $y = f^{-1}(x) = -\sqrt{x}$ ON $[0, \infty)$



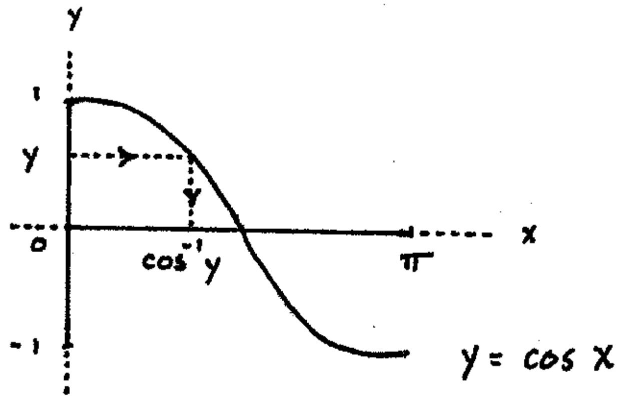
3. $y = \sin x$ ON $[-\frac{\pi}{2}, \frac{\pi}{2}]$

   $y = \sin^{-1} x = \arcsin x$ ON $[-1, 1]$

   = THE "ANGLE" IN $[-\frac{\pi}{2}, \frac{\pi}{2}]$ WHOSE SINE IS x

   (<u>NO</u> <u>SIMPLE</u> <u>FORMULA</u> <u>FOR</u> <u>COMPUTING</u> <u>THIS</u> <u>ONE</u>)

4. $y = \cos x$ on $[0, \pi]$

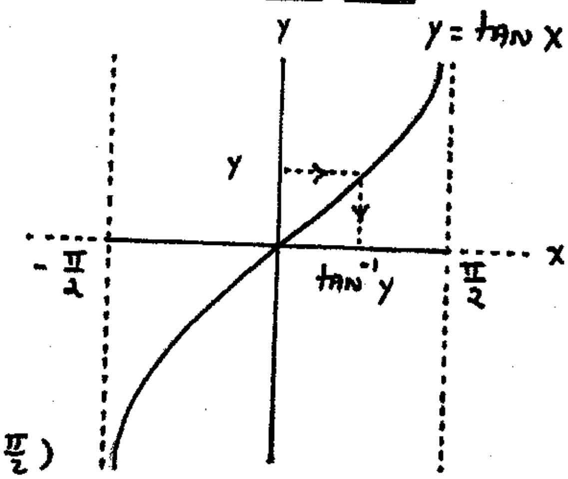


$y = \cos^{-1} x = \arccos x$ on $[-1, 1]$

= THE "ANGLE" IN $[0, \pi]$ WHOSE COSINE IS x

(NO SIMPLE FORMULA FOR COMPUTING THIS ONE)

5. $y = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$



$y = \tan^{-1} x = \arctan x$ on $(-\infty, \infty)$

= THE "ANGLE" IN $(-\frac{\pi}{2}, \frac{\pi}{2})$ WHOSE TANGENT IS x

(NO SIMPLE FORMULA FOR COMPUTING THIS ONE)

ONE CAN DO SOMETHING SIMILAR FOR $\cot x$, $\sec x$ AND $\csc x$, BUT WE WILL HAVE RELATIVELY LITTLE USE FOR THESE.

NEW FUNCTIONS FROM OLD :

SUPPOSE f AND g ARE FUNCTIONS. IF x IS IN THE DOMAIN OF BOTH f AND g, THEN WE CAN DEFINE

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x).$$

IF x IS IN THE DOMAIN OF BOTH f AND g AND IF $g(x) \neq 0$, THEN WE CAN DEFINE

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} .$$

IF x IS IN THE DOMAIN OF g AND IF $g(x)$ IS IN THE DOMAIN OF f, THEN WE CAN DEFINE

$$(f \circ g)(x) = f(g(x)) .$$

EXAMPLE : $g(x) = \arcsin x$: DOMAIN IS $[-1,1]$ AND, FOR EVERY x IN $[-1,1]$, $\arcsin x$ IS IN $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$f(x) = \cos x$: DOMAIN IS ALL REAL NUMBERS.

THEN, FOR ANY x IN $[-1,1]$,

$$(f \circ g)(x) = f(g(x)) = \cos(\arcsin x)$$
$$= \sqrt{\cos^2(\arcsin x)}$$ BECAUSE THE COSINE IS ≥ 0 ON $[-\frac{\pi}{2}, \frac{\pi}{2}]$
$$= \sqrt{1 - \sin^2(\arcsin x)}$$
$$= \sqrt{1 - x^2}$$