MATH 680 LECTURE 26 RICCI AND SCALAR CURVATURES

<u>SUMMARY</u> <u>OF</u> <u>NOTATION</u>: (M, g) RIEMANNIAN n-MANIFOLD

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

$$[X,Y,Z,W] = g(R(X,Y)Z, W)$$

$$R(\partial_i, \partial_j)\partial_k = R^{\ell}_{ijk}\, \partial_\ell$$

$$[\partial_i, \partial_j, \partial_k, \partial_\ell] = R_{ijk\ell} = R^{a}_{ijk}\, g_{a\ell}$$

$x, y, z, w \in T_p(M)$: $[x,y,z,w] = [X,Y,Z,W](p)$
FOR ANY X, Y, Z, W WITH
$X(p) = x$, $Y(p) = y$, $Z(p) = z$, $W(p) = w$

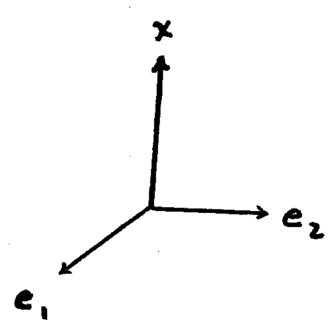
$$K(\sigma) = K(x,y) = \frac{[x,y,x,y]}{g(x,x)g(y,y) - (g(x,y))^2}$$

K CAPTURES ALL OF THE INFORMATION IN R, BUT IS COMPLICATED.

NOW TRY TO CAPTURE JUST SOME OF THIS INFORMATION BY "AVERAGING":

<u>RICCI</u> <u>CURVATURE</u> (<u>MOTIVATION</u>):

FIX $p \in M$. FOR ANY $x \in T_p(M)$, $x \neq 0$, CHOOSE A g-ORTHONORMAL BASIS $\{e_1, \ldots, e_{n-1}\}$ FOR THE ORTHOGONAL COMPLEMENT $x^\perp$:



THIS DETERMINES $n-1$ 2-DIMENSIONAL PLANES

$$\sigma_1 = \text{SPAN}\{x, e_1\} \quad , \dots , \quad \sigma_{n-1} = \text{SPAN}\{x, e_{n-1}\} .$$

EACH HAS A SECTIONAL CURVATURE AT p:

$$K(\sigma_i) = K(x, e_i) = \frac{[x, e_i, x, e_i]}{g(x,x)g(e_i, e_i) - (g(x, e_i))^2}$$

$$= \frac{[x, e_i, x, e_i]}{g(x,x)}$$

$$= \frac{g(R(x, e_i)x, e_i)}{g(x,x)}$$

SINCE THE 2-PLANE σ_i DOES NOT CHANGE WE MIGHT AS WELL ASSUME x IS A <u>UNIT</u> <u>VECTOR</u>, IN WHICH CASE

$$K(\sigma_i) = g(R(x, e_i)x, e_i)$$

THE AVERAGE OF THESE $n-1$ SECTIONAL CURVATURES AT p IS $\frac{1}{n-1}$ TIMES

$$\sum_{i=1}^{n-1} g(R(x, e_i)x, e_i)$$

AND, UNLIKE EACH OF THE TERMS $g(R(x, e_i)x, e_i)$ INDIVIDUALLY, THIS <u>DOES</u> <u>NOT</u> <u>DEPEND</u> <u>ON</u> <u>THE</u> <u>CHOICE</u> <u>OF</u> <u>THE</u> <u>ORTHONORMAL</u> <u>BASIS</u>, AS WE NOW PROVE.

THE WAY WE WILL PROVE THIS IS A BIT INDIRECT (BUT VERY CUTE) SO LET ME GIVE YOU A CLUE :

> A LINEAR TRANSFORMATION ON A VECTOR SPACE HAS A MATRIX RELATIVE TO ANY CHOICE OF BASIS. THE MATRIX DEPENDS ON WHICH BASIS YOU CHOOSE, BUT ITS TRACE DOES NOT.

STILL WITH $p \in M$ FIXED, LET US ALSO FIX $x, y \in T_p(M)$. DEFINE A LINEAR TRANSFORMATION

$$T_p(M) \longrightarrow T_p(M)$$

$$z \longrightarrow R(x,z)y \, .$$

WE WILL DENOTE THE TRACE OF THIS LINEAR TRANSFORMATION (WHICH IS ON $T_p(M)$ AND DEPENDS ON x, y)

$$\mathrm{Ric}_p(x,y) = \mathrm{trace}\,\big(z \to R(x,z)y \big)$$

AND NOW WE WILL COMPUTE IT. LET $\{ e_1, \dots, e_{n-1}, e_n \}$ BE A g-ORTHONORMAL BASIS FOR $T_p(M)$ WITH $x \in \text{SPAN}\{e_n\}$ AND $x^\perp = \text{SPAN}\{e_1, \dots, e_{n-1}\}$. THE MATRIX OF $z \to R(x,z)y$ IS OBTAINED AS FOLLOWS :

$$e_i \longrightarrow R(x,e_i)y = \sum_{j=1}^{n} g(R(x,e_i)y, e_j)\, e_j$$

MATRIX OF $z \to R(x,z)y$: $\quad \big(g(R(x,e_i)y, e_j) \big)_{i,j=1,\dots,n}$

THUS, THE TRACE IS

$$\mathrm{Ric}_p(x,y) = \sum_{i=1}^{n} g(R(x,e_i)y, e_i)$$

$$= \sum_{i=1}^{n-1} g(R(x,e_i)y, e_i)$$

SINCE x PARALLEL TO e_n IMPLIES

$g(R(x,e_n)y, e_n) = [x, e_n, y, e_n] = 0$

(SKEW-SYMMETRY IN THE FIRST TWO SLOTS).

SINCE THIS IS INDEPENDENT OF THE CHOICE OF ORTHONORMAL BASIS, SO IS $\sum_{i=1}^{n-1} g(R(x,e_i)x, e_i)$, AS PROMISED.

WITH ALL OF THIS AS MOTIVATION WE ARRIVE AT THE MODERN, COORDINATE-FREE DEFINITION:

(M, g) A RIEMANNIAN n-MANIFOLD

$p \in M$

DEFINE A BILINEAR FORM

$$\mathrm{Ric}_p : T_p(M) \times T_p(M) \to \mathbb{R}$$

BY

$$\mathrm{Ric}_p(x,y) = \mathrm{trace}\ (z \to R(x,z)y)$$

THIS IS CALLED THE <u>RICCI</u> <u>TENSOR</u> <u>AT</u> p

IF $\{e_1, \dots, e_{n-1}, e_n\}$ IS ANY ORTHONORMAL BASIS FOR $T_p(M)$, THEN

$$\begin{aligned} Ric_p(x,y) &= \sum_{i=1}^{n} g(R(x,e_i)y, e_i) \\ &= \sum_{i=1}^{n} [x, e_i, y, e_i] \end{aligned}$$

<u>NOTE</u>: FROM THIS AND THE SYMMETRY OF $[\ ,\ ,\ ,\ ]$ IN THE FIRST AND SECOND PAIRS IT IS CLEAR THAT Ric_p IS A <u>SYMMETRIC</u> BILINEAR FORM:

$$Ric_p(y,x) = Ric_p(x,y)$$

IF $x \in T_p(M)$ IS A UNIT VECTOR WE DEFINE

$$Ric_p(x) = Ric_p(x,x)$$

= <u>RICCI</u> <u>CURVATURE</u> <u>AT</u> p <u>IN</u> <u>THE</u> <u>DIRECTION</u> x

$$= \sum_{i=1}^{n-1} g(R(x,e_i)x, e_i)$$

WHERE $\{e_1, \dots, e_{n-1}\}$ IS AN ORTHONORMAL BASIS FOR $x^{\perp}$ IN $T_p(M)$.

= $(n-1)$ TIMES THE AVERAGE OF THE SECTIONAL CURVATURES OF THE PLANES $\sigma_i = \text{SPAN}\{x, e_i\}$

FINALLY, WE DEFINE

$$Ric : \mathcal{T}(TM) \times \mathcal{T}(TM) \to C^{\infty}(M)$$

BY

$$Ric(X, Y)(p) = Ric_p(X(p), Y(p))$$

IF $\{E_1, \ldots, E_n\}$ IS A <u>LOCAL</u> <u>ORTHONORMAL</u> <u>FRAME</u> (I.E., A SET OF n SMOOTH VECTOR FIELDS ON SOME OPEN SET IN M WITH $\{E_1(p), \ldots, E_n(p)\}$ AN ORTHONORMAL BASIS FOR $T_p(M)$ AT EACH p IN THIS OPEN SET), THEN

$$Ric(X, Y) = \sum_{i=1}^{n} g(R(X, E_i)Y, E_i)$$

THUS, IN LOCAL COORDINATES,

$$\begin{aligned} Ric(X, Y) &= Ric(x^i \partial_i, y^j \partial_j) \\ &= Ric(\partial_i, \partial_j) x^i y^j \\ &= R_{ij} x^i y^j \end{aligned}$$

WHERE WE COMPUTE THE COMPONENTS R_{ij} AS FOLLOWS: AT EACH POINT,

$$Ric(\partial_i, \partial_j) = \mathrm{trace}(z \to R(\partial_i, z)\partial_j)$$

AND WE CAN COMPUTE THE TRACE IN ANY BASIS FOR THE TANGENT SPACE, E.G., IN THE COORDINATE BASIS:

$$\partial_k \longrightarrow R(\partial_i, \partial_k)\partial_j = R^{\ell}_{\;ikj}\,\partial_\ell$$

MATRIX OF THE TRANSFORMATION IS

$$\left(R^{\ell}_{\;ikj} \right)_{k,\ell = 1,\dots,n}$$

SO THE TRACE IS

$$R^{k}_{\;ikj} \qquad (\text{SUM OVER } k = 1,\dots,n)$$

AND SO

$$R_{ij} = R^{k}_{\;ikj}$$

(IN OLD-FASHIONED BOOKS THIS LOCAL COMPONENT FORMULA IS OFTEN TAKEN AS THE DEFINITION OF THE RICCI TENSOR).

THUS, LOCALLY

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}(x^i\partial_i, y^j\partial_j) = R^{k}_{\;ikj}\, x^i y^j$$

<u>NOTE</u>: IN MANY WAYS THE RICCI TENSOR Ric IS ANALOGOUS TO THE METRIC TENSOR g (SYMMETRIC BILINEAR FORMS ON PAIRS OF TANGENT VECTORS). IT CAN HAPPEN THAT THEY ARE MORE THAN JUST ANALOGOUS: AN n-DIMENSIONAL RIEMANNIAN

MANIFOLD (M, g) IS CALLED AN <u>EINSTEIN MANIFOLD</u> IF

$$\mathrm{Ric} = \lambda g$$

FOR SOME CONSTANT λ (IF $n \geq 3$ IT IS ENOUGH TO ASSUME $\lambda \in C^\infty(M)$ AND IT FOLLOWS THAT λ MUST BE CONSTANT).

<u>REASON FOR THE TERMINOLOGY</u>: IF $n = 4$ AND g IS A LORENTZIAN (RATHER THAN RIEMANNIAN) METRIC, $\mathrm{Ric} = \lambda g$ IS THE EMPTY SPACE EINSTEIN FIELD EQUATIONS WITH A COSMOLOGICAL CONSTANT. WITHOUT A COSMOLOGICAL CONSTANT THEY ARE $\mathrm{Ric} = 0$.

<u>EXERCISE</u>: SHOW THAT IF (M, g) HAS CONSTANT CURVATURE, THEN IT IS EINSTEIN. MORE SPECIFICALLY, IF $\dim M = n$ AND $K \equiv K_0$, THEN

$$\mathrm{Ric} = K_0 (n-1) g$$

<u>HINT</u>: WE HAVE SHOWN THAT

$$[X, Y, Z, W] = K_0 \left(g(X,Z) g(Y,W) - g(Y,Z) g(X,W) \right)$$

SO

$$R_{ijk\ell} = K_0 (g_{ik} g_{j\ell} - g_{jk} g_{i\ell}).$$

NOW JUST COMPUTE $R_{ik} = R^{j}_{\ ijk} = g^{aj} R_{ijka}$.

EXERCISE : SHOW THAT FOR A 2-DIMENSIONAL RIEMANNIAN MANIFOLD (M, g),

$$\mathrm{Ric} = K g$$

WHERE K IS THE GAUSSIAN CURVATURE.

THUS, A RIEMANNIAN 2-MANIFOLD IS EINSTEIN IF AND ONLY IF IT HAS CONSTANT SECTIONAL (I.E., GAUSSIAN) CURVATURE. THE SAME IS TRUE (BUT NOT OBVIOUS) FOR RIEMANNIAN 3-MANIFOLDS.

THE RICCI CURVATURE CONTAINS SOME, BUT NOT ALL OF THE INFORMATION IN THE SECTIONAL / RIEMANN CURVATURE (THE REST IS CONTAINED IN SOMETHING CALLED THE "WEYL TENSOR", WHICH WE WILL NOT DISCUSS).

NOW WE TURN TO AN OBJECT THAT CONTAINS EVEN LESS INFORMATION, BUT IS CORRESPONDINGLY EASIER TO RELATE TO :

SCALAR CURVATURE (MOTIVATION):

WE HAVE DEFINED, FOR EVERY $X \in T_p(M)$, THE RICCI CURVATURE $Ric_p(X)$ OF M AT p IN THE DIRECTION X AS ($n-1$ TIMES) AN AVERAGE OF SECTIONAL CURVATURES AT p.

THE IDEA BEHIND DEFINING THE SCALAR CURVATURE $S(p)$ AT p IS TO AVERAGE ONCE MORE, THIS TIME AVERAGING THE RICCI CURVATURES IN n ORTHOGONAL DIRECTIONS AT p.

THUS, ONE WOULD CHOOSE AN ORTHONORMAL BASIS $\{e_1, \ldots, e_n\}$ FOR $T_p(M)$ AND COMPUTE

$$\frac{1}{n} \sum_{i=1}^{n} Ric_p(e_i)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j \neq i} g(R(e_i, e_j)e_i, e_j)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} g(R(e_i, e_j)e_i, e_j)$$

SINCE $g(R(e_i, e_i)e_i, e_i) = 0$
SO THERE IS NO HARM IN INCLUDING IT

$$= \frac{1}{n} \sum_{i,j=1}^{n} g(R(e_i, e_j)e_i, e_j)$$

AS USUAL, ONE NEEDS TO KNOW THAT THIS DOES NOT DEPEND ON THE CHOICE OF ORTHONORMAL BASIS SO WE WILL EXHIBIT AN INVARIANT

FORMULATION (AGAIN AS THE TRACE OF A LINEAR TRANSFORMATION).

> WE WILL USE THE FOLLOWING CRITICAL FACT FROM LINEAR ALGEBRA: LET V BE A FINITE-DIMENSIONAL REAL VECTOR SPACE WITH AN INNER PRODUCT $\langle\ ,\ \rangle$ DEFINED ON IT. LET V^* BE THE DUAL SPACE (SPACE OF REAL-VALUED LINEAR FUNCTIONS ON V). THEN $\forall\, \varphi \in V^*$ THERE IS A UNIQUE $v \in V$ SUCH THAT
>
> $$\varphi(y) = \langle v, y \rangle$$
>
> $\forall\, y \in V$ (EVERY LINEAR FUNCTIONAL ON V IS "DOTTING WITH" SOME UNIQUE VECTOR IN V).

NOW, $Ric_p(x,y)$ IS A REAL-VALUED BILINEAR FORM ON $T_p(M)$. FOR EACH FIXED $x \in T_p(M)$,

$$y \longrightarrow Ric_p(x,y)$$

IS THEREFORE A REAL-VALUED LINEAR FUNCTIONAL ON $T_p(M)$.

SINCE $T_p(M)$ HAS AN INNER PRODUCT (NAMELY, g_p) THIS LINEAR FUNCTIONAL IS DOTTING WITH SOMETHING IN $T_p(M)$. THIS "SOMETHING" WILL DEPEND ON x SO WE DENOTE IT $S(x)$:

$$Ric_p(x, y) = g_p(S(x), y)$$

$\forall\ x, y \in T_p(M)$. CLEARLY,

$$S : T_p(X) \to T_p(X)$$

IS A LINEAR TRANSFORMATION AND IT IS, MOREOVER, SELF-ADJOINT WITH RESPECT TO THE INNER PRODUCT g_p

$$g_p(S(x), y) = g_p(x, S(y))$$

SINCE $g_p(S(x), y) = Ric_p(x, y) = Ric_p(y, x) = g_p(S(y), x) = g_p(x, S(y))$.

THE OBJECT WE WANT IS THE TRACE OF THE LINEAR TRANSFORMATION S. SINCE WE CAN COMPUTE IT IN ANY BASIS WE WILL USE AN ORTHONORMAL ONE : $\{e_1, \dots, e_n\}$

MATRIX OF $S : T_p(X) \to T_p(X)$:

$$S(e_i) = \sum_{j=1}^{n} g_p(S(e_i), e_j)e_j$$

MATRIX : $\left(g_p(S(e_i), e_j) \right)_{i,j=1,\dots,n}$

TRACE : $\displaystyle\sum_{i=1}^{n} g_p(S(e_i), e_i) = \sum_{i=1}^{n} Ric_p(e_i, e_i)$

$$= \sum_{i=1}^{n} Ric_p(e_i)$$

$$= \sum_{i,j=1}^{n} g(R(e_i, e_j)e_i, e_j)$$

THUS, THE AVERAGE OF RICCI CURVATURES IN THE n ORTHOGONAL DIRECTIONS $e_1, \dots, e_n$ AT p IS JUST $\frac{1}{n}$ TIMES THE TRACE OF THE LINEAR TRANSFORMATION

$$S : T_p(M) \to T_p(M)$$

DEFINED BY

$$Ric_p(x, y) = g_p(S(x), y)$$

$\forall\, y \in T_p(M)$ AND SO DOES NOT DEPEND ON THE CHOICE OF ORTHONORMAL BASIS $\{e_1, \dots, e_n\}$ FOR $T_p(M)$.

THIS LEADS US TO OUR DEFINITION :

THE <u>SCALAR CURVATURE</u> OF M AT p IS THE REAL NUMBER $S(p)$ DEFINED TO BE THE TRACE OF THE LINEAR TRANSFORMATION $S : T_p(M) \to T_p(M)$ DEFINED BY

$$g_p(S(x), y) = Ric_p(x, y)$$

$\forall\, y \in T_p(M)$. IF $\{e_1, \dots, e_n\}$ IS AN ORTHONORMAL BASIS FOR $T_p(M)$, THEN

$$S(p) = \sum_{i=1}^{n} Ric_p(e_i)$$

$$= \sum_{i,j=1}^{n} g(R(e_i, e_j)e_i, e_j)$$

SINCE A TRACE IS INVARIANT UNDER CHANGE OF BASIS WE CAN ALSO COMPUTE THIS IN A COORDINATE BASIS TO GET LOCAL COORDINATE EXPRESSIONS FOR S. I'll LEAVE THIS FOR YOU.

<u>EXERCISE</u> : FIND, AT ANY POINT p, THE MATRIX OF $S : T_p(M) \to T_p(M)$ RELATIVE TO A LOCAL COORDINATE BASIS $\{ \partial_1(p), \dots, \partial_n(p) \}$ AND COMPUTE ITS TRACE TO OBTAIN

$$S = g^{ij} R_{ij}$$

<u>EXERCISE</u> : SHOW THAT IF (M, g) HAS CONSTANT CURVATURE K_0, THEN

$$S(p) = n(n-1) K_0$$

$\forall\, p \in M \quad (n = \dim M)$.

<u>EXERCISE</u> : SHOW THAT IF $\dim M = 2$, THEN

$$S = 2\mathcal{K}$$

WHERE $\mathcal{K}$ IS THE GAUSSIAN CURVATURE.

THE FOLLOWING FEW PAGES GIVE A QUICK SUMMARY OF SOME "GLOBAL RIEMANNIAN GEOMETRY" THAT WE WILL NOT GET TO (FROM RIEMANNIAN GEOMETRY AND GEOMETRIC ANALYSIS, JÜRGEN JOST, SPRINGER-VERLAG, 2002).

A Short Survey on Curvature and Topology

We have now covered half of the chapters of the present textbook and the more elementary aspects of the subject. Before penetrating into more advanced topics, a short survey on some directions of global Riemannian geometry may be a useful orientation guide. Because of the size and scope of the present book, this survey needs to be selective.

A basic question, formulated in particular by H. Hopf, is to what extent the existence of a Riemannian metric with particular curvature properties restricts the topology of the underlying differentiable manifold.

The classical example is the

Gauss-Bonnet Theorem. *Let M be a compact oriented, two-dimensional Riemannian manifold with curvature K. Then its Euler characteristic is determined by*

$$\chi(M) = \frac{1}{2\pi} \int_M K \,\mathrm{dvol}\, M.$$

We have also seen some higher dimensional examples already, namely the Theorem 4.1.2 of Synge on manifolds with positive sectional curvature, the Theorem 3.5.1 of Bochner and the Bonnet-Myers Theorem (Corollary 4.3.1) on manifolds of positive Ricci curvature. We have already seen a result for nonpositive sectional curvature, namely the Hadamard-Cartan Theorem (Corollary 4.8.1) that a simply-connected, complete manifold of nonpositive sectional curvature is diffeomorphic to some $\mathbb{R}^n$, and in Chapter 8, we shall prove the Preissmann Theorem (Corollary 8.10.2) that any abelian subgroup of the fundamental group of a compact manifold of negative sectional curvature is infinite cyclic, i.e. isomorphic to $\mathbb{Z}$. In order to put these results in a better perspective, we want to discuss the known implications of curvature properties for the topology more systematically.

We start with the implications of positive sectional curvature. Here, we have the

Sphere Theorem. *Let M be a compact, simply connected Riemannian manifold whose sectional curvature K satisfies*

$$0 < \frac{1}{4}\kappa < K \leq \kappa$$

for some fixed number κ. Then M is homeomorphic to the sphere S^n ($n = \dim M$).

This was shown by M. Berger, Les variétés Riemanniennes $(\frac{1}{4})$-pincées, Ann. Scuola Norm. Sup. Pisa, Ser. III, 14 (1960), 161-170, and W. Klingenberg, Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung, Comm. Math. Helv. 35 (1961), 47-54.

The pinching number $\frac{1}{4}$ is optimal in even dimensions ≥ 4, because $\mathbb{CP}^m$ (see § 5.1) is simply connected, has sectional curvature between $\frac{1}{4}$ and 1 for its Fubini-Study metric and is not homeomorphic to S^{2m} for $m > 1$. In odd dimensions, the pinching number can be decreased below $\frac{1}{4}$, as shown by U. Abresch and W. Meyer, Pinching below $\frac{1}{4}$, injectivity radius, and conjugate radius, J.Diff.Geom. 40, (1994) 643-691, but the optimal value of the pinching constant is unknown at present.

For $n = 2$ or 3, the conclusion is valid already if M has positive sectional curvature. For $n = 2$, this follows from the Gauss-Bonnet Theorem. For $n = 3$, R. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17 (1982), 255-306, showed that any simply connected compact manifold of positive Ricci curvature is diffeomorphic to S^3. Hamilton studied the so-called Ricci flow, i.e. he considered the evolution problem for a time dependent family of metrics g_{ij} on M with Ricci curvature R_{ij}.

$$\frac{\partial}{\partial t} g_{ij}(x,t) = \frac{2}{n} r(t) g_{ij}(x,t) - 2R_{ij}(x,t),$$

with initial metric $g_{ij}(x,0) = g^0_{ij}(x)$ where

$$r(t) = \frac{\int R(x,t)\,\mathrm{dvol}(g(\cdot,t))}{\int \mathrm{dvol}(g(\cdot,t))}$$

is the average of the scalar curvature of the metric $g_{ij}(\cdot,t)$. He showed that if g^0_{ij} is a metric with positive Ricci curvature on a compact 3-manifold, then a solution of this evolution problem exists for all time, the Ricci curvature stays positive for all t, and as $t \to \infty$, $g_{ij}(\cdot,t)$ converges to a metric of constant (positive) sectional curvature.

This method has since become important in Riemannian geometry, although in general without suitable curvature assumptions on the initial metric, singularities will develop in finite time, and these singularities still await a thorough understanding.

It is not known whether M as in the sphere theorem is diffeomorphic instead of just homeomorphic to S^n. In other words, one has to exclude that exotic spheres carry $\frac{1}{4}$-pinched metrics. This so far has only been achieved for certain pinching numbers greater than $\frac{1}{4}$; for a sample of results see e.g. H. Im Hof and E. Ruh, An equivariant pinching theorem, Comm. Math. Helv. 50 (1975), 389-401; E. Ruh, Riemannian manifolds with bounded curvature ratios, J. Diff. Geom. 17 (1982), 643-653; G. Huisken, Ricci deformation of the metric on a Riemannian manifold, J. Diff. Geom. 18 (1985), 47-62, and Y. Sugama, Differentiable sphere theorem by curvature pinching, Tohoku Math. J. 47, (1995), 15-29.

It is not even known whether some exotic spheres can carry a metric of positive sectional curvature. Also, the problem of H. Hopf whether $S^2 \times S^2$ can carry a metric of positive sectional curvature is unsolved. The essential question is to understand compact, simply connected Riemannian manifolds of positive sectional curvature. Only very few examples of such manifolds are known. In fact, besides the general series of compact rank one symmetric spaces (spheres, complex projective spaces (see § 5.1 below) in all even dimensions, quaternionic projective spaces in all dimensions that are multiples of 4, and the Cayley projective plane in dimension 16), one only knows the family of Allof-Wallach spaces in dimension 7 and the isolated examples of Eschenburg and Bazaikin.

In recent years, however, the first indications of a general structure theory seem to emerge, in the work of A. Petrunin, X. Rong, W. Tuschmann, Collapsing vs. positive pinching, GAFA 9 (1999), 699-735, A. Petrunin, W. Tuschmann, Diffeomorphism finiteness, positive pinching, and second homotopy, GAFA 9 (1999), 736-774. F. Fang, X. Rong, Positive pinching, volume and second Betti number, GAFA 9 (1999), 641-674. For a comprehensive treatment, see W. Tuschmann, Endlichkeitssätze und positive Krümmung, Habilitation thesis, Leipzig, 2000. Essential points of this approach are that one studies the more general class of Alexandrov spaces of positive curvature which allows to study sequences of positively curved spaces and use compactness arguments by the result of Nikolaev quoted below, and in particular to utilize collapsing techniques and that the rôle of the second homotopy group becomes more prominent in determining the topological possibilities of positively curved spaces. (So, one might speculate that the theory of minimal 2-spheres developed in § 8.3 might furnish useful tools for understanding the topology of positively curved spaces.)

We also mention that B. Wilking, Manifolds with positive sectional curvature almost everywhere, Preprint, 2000, showed that in general, a metric of positive curvature outside a finite number of points on a compact manifold cannot be deformed into a metric of positive curvature everywhere.

For positive Ricci curvature, we have already exhibited some results. An important generalization of these results is Gromov's

First Betti Number Theorem. *Let M be a compact Riemannian manifold of dimension n, with diameter $\leq D$ and Ricci curvature $\geq \lambda$ (i.e. $(R_{ij} - \lambda g_{ij})_{i,j}$ is a positive semidefinite tensor). Then the first Betti number satisfies*

$$b_1(M) \leq f(n, \lambda, D)$$

with an explicit function $f(n, \lambda, D)$

$$(f(n,0,D) = n, \quad f(n,\lambda,D) = 0 \quad \textit{for } \lambda > 0).$$

See M. Gromov, Structures métriques pour les variétés Riemanniennes, rédigé par J. Lafontaine et P. Pansu, Cedic-Nathan, Paris, 1980.

Finally, it has been determined which simply connected manifolds admit metrics of positive scalar curvature and which ones don't, in the work of R. Schoen and S.T. Yau, The structure of manifolds with positive scalar curvature, Man. math. 28 (1979), 159-183; M. Gromov and B. Lawson, The classification of simply connected manifolds of positive scalar curvature, Ann. Math. 111 (1980), 423-434 and S. Stolz, Simply connected manifolds of positive scalar curvature, Ann. Math. 136 (1992), 511-540.

In the non simply-connected case, also restrictions for positive scalar curvature are known. For example, for dimension ≤ 7, a torus cannot admit a metric of positive scalar curvature, see R. Schoen and S.T. Yau, Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with nonnegative scalar curvature, Ann. Math. 110 (1979), 127-142. Such a result for any n and other restrictions on metrics of positive scalar curvature were given by M. Gromov and B. Lawson, Spin and scalar curvature in the presence of a fundamental group, Ann. Math. 111 (1980), 209-230.

The preceding results all apply to compact manifolds. For noncompact manifolds, let us only quote the

Splitting Theorem. *The universal covering $\tilde{M}$ of a compact Riemannian manifold with nonnegative Ricci curvature splits isometrically as a product $\tilde{M} = N \times \mathbb{R}^k, 0 \leq k \leq \dim M$, where N is a compact manifold.*

See J. Cheeger and D. Gromoll, The splitting theorems for manifolds of nonnegative Ricci curvature, J. Diff. Geom. 6 (1971), 119-129.

For a more detailed survey of manifolds of nonnegative curvature, we refer to the survey article D. Gromoll, Spaces of nonnegative curvature, Proc. Sym. Pure Math. 54 (1993), Part 3, 337-356.

For manifolds of negative or nonpositive sectional curvature, much more is known than for those of positive curvature. Some discussion can be found in the Perspectives on 8.10. We also refer to the survey article, P. Eberlein, U. Hamenstädt and V. Schroeder, Manifolds of nonpositive curvature, Proc. Symp. Pure Math. 54 (1993), Part 3, 179-227.

J. Lohkamp, Negatively Ricci curved manifolds, Bull. AMS 27 (1992), 288-292, and Metrics of negative Ricci curvature, Ann. Math. 140 (1994) 655-683, proved that any differentiable manifold of dimension ≥ 3 admits a complete metric of negative Ricci curvature. As a consequence, negative Ricci curvature does not imply any topological restrictions.

Riemannian manifolds of vanishing sectional curvature are called flat. The compact ones are classified by the

Bieberbach Theorem. *Let M be a compact flat Riemannian manifold of dimension n. Then its fundamental group contains a free abelian normal subgroup of rank n and finite index. Thus, M is a finite quotient of a flat torus.*

In analogy to the sphere theorem, one may ask about the structure of Riemannian manifolds that are almost flat in the sense that their curvature is close to zero. Since the curvature of a Riemannian metric may always be made arbitrarily small by rescaling the metric, the appropriate curvature condition has to be more carefully formulated in a scaling invariant manner. Let us look at the typical example:

We consider the nilpotent Lie group H of upper triangular matrices with 1's on the diagonal. Its Lie algebra is

$$\mathfrak{h} = \left\{A = \begin{pmatrix} 0 & & a_{ij} \\ & \vdots & \\ 0 & & 0 \end{pmatrix} : a_{ij} \in \mathbb{R}, 1 \leq i < j \leq n\right\}.$$

On $\mathfrak{h}$, we may introduce a family of scalar products via

$$\|A\|_q^2 := \sum_{i<j} a_{ij}^2 q^{2(j-i)}$$

for $q > 0$. These scalar products induce left invariant Riemannian metrics on H whose curvature can be estimated as

$$\|R_q(A, B) \subset \|_q \leq 24(n-2)^2 \|A\|_q^2 \|B\|_q^2 \|C\|_q^2.$$

This bound is independent of q. By a q-independent rescaling, we may therefore assume that the sectional curvature satisfies $|K| \leq 1$. We let $H(\mathbb{Z})$ be the subgroup of H with integer entries, and one may thus construct left invariant metrics on H which induce on the quotient $H/H(\mathbb{Z})$ metrics with $|K| \leq 1$ and diam $< \varepsilon$, for every $\varepsilon > 0$, simply by choosing q sufficiently small.

Conversely,

Theorem. *For every n, there exists $\varepsilon(n) > 0$ with the property that any compact n-dimensional Riemannian manifold M with*

$$|K| \, (\mathrm{diam})^2 < \varepsilon(n)$$

is diffeomorphic to a finite quotient of a nilmanifold. (A nilmanifold is by definition a compact homogeneous space of a nilpotent Lie group.)

This is due to M. Gromov, see P. Buser and H. Karcher, Gromov's almost flat manifolds, Astérisque 81, 1981, for an exposition, and for the refinement that M as above is actually an infranilmanifold by E. Ruh, Almost flat manifolds, J. Diff. Geom. 17 (1982), 1-14.

In order to place this result in a broader context, we introduce the notions of convergence and collapse of manifolds. For compact subsets A_1, A_2 of a metric space Z, we define

$$d_H^Z(A_1, A_2) := \inf\{r : A_1 \subset \bigcup_{x\in A_2} \overset{\circ}{B}(x,r), \quad A_2 \subset \bigcup_{x\in A_1} \overset{\circ}{B}(x,r)\}$$

where $\overset{\circ}{B}(x,r) := \{y \in Z : d(x,y) < r\}$.

For compact metric spaces X_1, X_2, their Hausdorff distance is

$$d_H(X_1, X_2) := \inf_Z \{d_H^Z(i(X_1), j(X_2)),$$
$$\text{where } i : X_1 \to Z, j : X_2 \to Z \text{ are}$$
$$\text{isometries into a metric space } Z\}.$$

This distance then defines the notion of Hausdorff convergence of compact metric spaces. Let M_0 be a compact differentiable manifold of dimension n. We say that M_0 admits a collapse to a compact metric space X of lower (Hausdorff) dimension than M_0 if there exists a sequence $(g_j)_{j\in\mathbb{N}}$ of Riemannian metrics with uniformly bounded curvature on M_0 such that the Riemannian manifolds (M_0, g_j) as metric spaces converge to X. This phenomenon has been introduced and studied by J. Cheeger and M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded, I. J. Diff. Geom. 23 (1983), 309-346; II, J. Diff. Geom. 32 (1990), 269-298 and K. Fukaya, A boundary of the set of Riemannian manifolds with bounded curvatures and diameters, J. Diff. Geom. 28 (1988), 1-21.

It is easy to see that any torus can collapse to a point; for this purpose, one just rescales a given flat metric by a factor ε and lets $\varepsilon \to 0$. The diameter then shrinks to 0, while the curvature always remains 0. Berger showed that S^3 admits a collapse onto S^2. The construction is based on the Hopf fibration $\pi : S^3 \to S^2 = \mathbb{CP}^1$ (see § 5.1), and one lets the fibers shrink to zero in length.

In this terminology the above theorem (as refined by Ruh) says that those manifolds that can collapse to a point are precisely the infranilmanifolds. More recently, it was shown by W. Tuschmann, Collapsing, solvmanifolds, and infrahomogenous spaces, Diff. Geom. Appl. 7 (1997), 251-264, that any manifold that admits a collapse onto some flat orbifold is homeomorphic to an infrasolvmanifold and conversely, that any infrasolvmanifold also admits a sequence of Riemannian metrics for which it collapses to a compact flat orbifold. Here, an infrasolvmanifold is a certain type of quotient of a solvable Lie group.

We next mention the following result of J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92 (1970), 61-74, with the improvements by S. Peters, Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds, J. reine angew. Math. 394 (1984), 77-82.

Finiteness Theorem. *For any $n \in \mathbb{N}$, $\Lambda < \infty$, $D < \infty, v > 0$, the class of compact differentiable manifolds of dimension n admitting a Riemannian metric with*

$$|K| \le \Lambda, \ \mathit{diam} \le D, \ \mathit{Volume} \ge v$$

consists of at most finitely many diffeomorphism types.

The lower positive uniform bound on volume prevents collapsing and is necessary for this result to hold.

Diffeomorphism finiteness can however actually also be obtained if no volume bounds are present and collapsing may take place.

This is demonstrated by the following recent finiteness theorem by A. Petrunin and W. Tuschmann, Diffeomorphism Finiteness, Positive Pinching, and Second Homotopy, GAFA 9 (1999), 736-774. Instead of volume bounds this result only uses a merely topological condition:

π_2-Finiteness Theorem. *For any $n \in \mathbb{N}$, $\Lambda < \infty$, and $D < \infty$, the class of compact simply connected differentiable manifolds of dimension n with finite second homotopy group admitting a Riemannian metric with*

$$|K| \ge \Lambda, \ \mathit{diam} \ \le D$$

consists of at most finitely many diffeomorphism types.

Cheeger's finiteness theorem was refined in the so-called Gromov convergence theorem, which we are going to present in the form proved by S. Peters, Convergence of Riemannian manifolds, Compos. Math. 62 (1987), 3-16 and R. Greene and H. Wu, Lipschitz convergence of Riemannian manifolds, Pacific J. Math. 131 (1988), 119-141.

Convergence Theorem. *Let $(M_j, g_j)_{j\in\mathbb{N}}$ be a sequence of Riemannian manifolds of dimension n satisfying the assumptions of the finiteness theorem with Λ, D, v independent of j. Then a subsequence converges in the Hausdorff distance and (after applying suitable diffeomorphisms) also in the (much stronger) $C^{1,\alpha}$ topology (for any $0 < \alpha < 1$) to a differentiable manifold with a $C^{1,\alpha}$-metric.*

Such a family of manifolds is known to have a uniform lower bound on their injectivity radius. The crucial ingredient in the proof then are the a-priori estimates of Jost-Karcher for harmonic coordinates described in the Perspectives on 8.10. Namely, these estimates imply convergence of subsequences of local coordinates on balls of fixed size, and the limits of these coordinates then are coordinates for the limiting manifold.

I.G. Nikolaev, Bounded curvature closure of the set of compact Riemannian manifolds, Bull. AMS 24 (1991), 171-177, showed that the Hausdorff limits of sequences of compact n-dimensional Riemannian manifolds of uniformly bounded curvature and diameter and with volume bounded away from 0 uniformly are precisely the smooth compact n-manifolds with metrics of bounded curvature in the sense of Alexandrov.

Let us conclude this short survey by listing some other textbooks on Riemannian geometry that treat various selected topics of global differential geometry and which complement the present book.

I. Chavel, Riemannian geometry – A modern introduction, Cambridge University Press, 1993.

J. Cheeger and D. Ebin, Comparison theorems in Riemannian geometry, North Holland, 1975.

M. do Carmo, Riemannian geometry, Birkhäuser, 1992.

S. Gallot, D. Hulin, and J. Lafontaine, Riemannian geometry, Springer, 1987.

D. Gromoll, W. Klingenberg, and W. Meyer, Riemannsche Geometrie im Großen, Springer LNM 55, ^{2}1975.

W. Klingenberg, Riemannian geometry, de Gruyter, 1982.

P. Petersen, Riemannian geometry, Springer, 1998.

T. Sakai, Riemannian geometry, Amer. Math. Soc., 1995.

Finally, we wish to mention the stimulating survey article

M. Berger, Riemannian geometry during the second half of the twentieth century, Jber. DMV 100 (1998), 45-208

5. Symmetric Spaces and Kähler Manifolds

5.1 Complex Projective Space. Definition of Kähler Manifolds

We consider the complex vector space $\mathbb{C}^{n+1}$. A complex linear subspace of $\mathbb{C}^{n+1}$ of complex dimension one is called a line. We define the complex projective space $\mathbb{CP}^n$ as the space of all lines in $\mathbb{C}^{n+1}$. Thus, $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ by the equivalence relation

$$Z \sim W :\iff \exists \lambda \in \mathbb{C}\backslash\{0\} : W = \lambda Z.$$

Namely, two points of $\mathbb{C}^{n+1}\backslash\{0\}$ are equivalent iff they are complex linearly dependent, i.e. lie on the same line. The equivalence class of Z is denoted by [Z] .

We also write

$$Z = (Z^0, \dots, Z^n) \in \mathbb{C}^{n+1}$$

and define

$$U_i := \{[Z] : Z^i \neq 0\} \subset \mathbb{CP}^n,$$

i.e. the space of all lines not contained in the complex hyperplane $\{Z^i = 0\}$. We then obtain a bijection

$$\varphi_i : U_i \to \mathbb{C}^n$$

via

$$\varphi_i([Z^0, \dots, Z^n]) := \left(\frac{Z^0}{Z^i}, \dots, \frac{Z^{i-1}}{Z^i}, \frac{Z^{i+1}}{Z^i}, \dots, \frac{Z^n}{Z^i}\right).$$

$\mathbb{CP}^n$ thus becomes a differentiable manifold, because the transition maps

$$\begin{aligned}
\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) = \{z = (z^1, \dots, z^n) \in \mathbb{C}^n : z^j \neq 0\} \to \varphi_j(U_i \cap U_j) \\
\varphi_j \circ \varphi_i^{-1}(z^1, \dots, z^n) = \varphi_j([z^1, \dots, z^i, 1, z^{i+1}, \dots, z^n]) \\
= \left(\frac{z^1}{z^j}, \dots, \frac{z^i}{z^j}, \frac{z^{i+1}}{z^j}, \dots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \dots, \frac{z^n}{z^j}\right) \\
(\text{w.l.o.g. } i < j)
\end{aligned}$$

are diffeomorphisms. They are even holomorphic; namely, with $z^k = x^k + iy^k$ $(i = \sqrt{-1})$ and