RIEMANNIAN MANIFOLDS

NOTE: THIS "LECTURE" IS INTENTIONALLY LONGER THAN PREVIOUS LECTURES IN THE HOPE THAT, BY THE END OF THE TERM, WE WILL HAVE SET ASIDE A BIT OF EXTRA TIME. IF NOT, WE WILL ADAPT AND MAKE SELECTIONS FROM THE FOLLOWING MATERIAL.

V = n-DIMENSIONAL REAL VECTOR SPACE

V^* = THE DUAL OF V = SET OF ALL REAL-VALUED LINEAR FUNCTIONS ON V

$$\alpha : V \to \mathbb{R}$$

$$\alpha(a_1 v_1 + a_2 v_2) = a_1 \alpha(v_1) + a_2 \alpha(v_2)$$

$$\forall\, a_1, a_2 \in \mathbb{R} \quad \forall\, v_1, v_2 \in V$$

VECTOR SPACE STRUCTURE OF V^* IS "POINTWISE", I.E., IF $\alpha, \beta \in V^*$, THEN $\alpha + \beta$ IS DEFINED BY

$$(\alpha+\beta)(v) = \alpha(v) + \beta(v) \qquad \forall\, v \in V$$

AND IF $a \in \mathbb{R}$,

$$(a\alpha)(v) = a(\alpha(v)) \qquad \forall\, v \in V.$$

IF $\{e_1, \dots, e_n\}$ IS A BASIS FOR V, THEN DEFINE $e^1, \dots, e^n$ IN V^* BY

$$e^i(e_j) = \delta^i_j$$

SO

$$\begin{aligned} e^i(v) &= e^i(v^1 e_1 + \dots + v^n e_n) \\ &= v^1 e^i(e_1) + \dots + v^n e^i(e_n) \\ &= v^i \end{aligned}$$

<u>EXERCISE</u> 92: SHOW THAT $e^1, \ldots, e^n$ ARE LINEARLY INDEPENDENT IN V^* AND THAT ANY $\alpha \in V^*$ CAN BE WRITTEN AS

$$\alpha = \alpha_i e^i \qquad \text{(SUMMATION CONVENTION)}$$

WHERE

$$\alpha_i = \alpha(e_i) \quad , \quad i = 1, \ldots, n.$$

THUS, $\{e^1, \ldots, e^n\}$ IS A BASIS FOR V^*, CALLED THE <u>DUAL BASIS</u> OF $\{e_1, \ldots, e_n\}$.

IN PARTICULAR,

$$\dim V^* = n = \dim V$$

SO V^* IS ISOMORPHIC TO V.

ELEMENTS OF V ARE CALLED <u>VECTORS</u> (OR <u>CONTRAVARIANT VECTORS</u>), WHILE ELEMENTS OF V^* ARE CALLED <u>COVECTORS</u> (OR <u>COVARIANT VECTORS</u>).

<u>EXAMPLE</u>: SUPPOSE X IS AN n-DIMENSIONAL SMOOTH MANIFOLD AND $p \in X$. THEN $T_p(X)$ IS AN n-DIMENSIONAL REAL VECTOR SPACE. ITS DUAL IS DENOTED

$$T_p^*(X),$$

CALLED THE <u>COTANGENT SPACE</u> AT p AND CONSISTS OF ALL LINEAR MAPS

$$\alpha : T_p(X) \longrightarrow \mathbb{R}.$$

HERE'S AN IMPORTANT WAY OF PRODUCING EXAMPLES OF COTANGENT VECTORS, I.E., ELEMENTS OF $T_p^*(X)$.

LET $f \in C^\infty(X)$. THIS GIVES RISE TO A MAP

$$T_p(X) \longrightarrow \mathbb{R}$$

WHICH SENDS

$$v_p \in T_p(X) \longrightarrow v_p(f) = \text{RATE OF CHANGE OF } f \text{ IN THE DIRECTION } v_p.$$

CLEARLY LINEAR SINCE

$$v_p + w_p \longrightarrow (v_p + w_p)(f) = v_p(f) + w_p(f)$$

AND

$$a v_p \longrightarrow (a v_p)(f) = a(v_p(f))$$

THUS, f DETERMINES AN ELEMENT OF $T_p^*(X)$. WE DENOTE THIS ELEMENT

$$df_p$$

AND CALL IT THE <u>DIFFERENTIAL</u> <u>OF</u> f <u>AT</u> p. THUS,

$$\boxed{df_p(v_p) = v_p(f)}$$

<u>EXERCISE</u> 93: LET (U, φ) BE A CHART ON X WITH $p \in U$ AND WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n \in C^\infty(U)$. THEN

$$\left\{ \frac{\partial}{\partial x^1}\Big|_p , \ldots , \frac{\partial}{\partial x^n}\Big|_p \right\}$$

IS A BASIS FOR $T_p(X)$. SHOW THAT

$$\{ dx^1_p , \ldots , dx^n_p \}$$

IS THE DUAL BASIS FOR $T_p^*(X)$.

IN PARTICULAR, ANY $\alpha \in T_p^*(X)$ CAN BE WRITTEN

$$\alpha = \alpha\left(\frac{\partial}{\partial x^i}\Big|_p \right) dx^i_p$$

SO, IF $\alpha = df$ FOR SOME $f \in C^\infty(X)$, THEN

$$df_p = df_p\left(\frac{\partial}{\partial x^i}\Big|_p \right) dx^i_p$$

$$= \frac{\partial f}{\partial x^i}(p)\, dx^i_p$$

DROPPING THE REFERENCES TO p WE GET THE FAMILIAR LOOKING

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n$$

NOW RETURN TO V WITH BASIS $\{e_1, \ldots, e_n\}$ AND V^* WITH THE DUAL BASIS $\{e^1, \ldots, e^n\}$.

SUPPOSE $\{\hat{e}_1, \ldots, \hat{e}_n\}$ IS ANOTHER BASIS FOR V AND $\{\hat{e}^1, \ldots, \hat{e}^n\}$ IS ITS DUAL BASIS FOR V^*.

WRITE

$$\begin{aligned} e_1 &= \Lambda^1{}_1 \hat{e}_1 + \cdots + \Lambda^n{}_1 \hat{e}_n = \Lambda^i{}_1 \hat{e}_i \\ &\vdots \\ e_n &= \Lambda^1{}_n \hat{e}_1 + \cdots + \Lambda^n{}_n \hat{e}_n = \Lambda^i{}_n \hat{e}_i \end{aligned}$$

OR, MORE BRIEFLY,

$$e_j = \Lambda^i{}_j \hat{e}_i \quad , \quad j = 1, \ldots, n$$

THINK OF THIS AS A MATRIX EQUATION :

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} \Lambda^1{}_1 & \cdots & \Lambda^n{}_1 \\ \vdots & & \vdots \\ \Lambda^1{}_n & \cdots & \Lambda^n{}_n \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{pmatrix}$$

THE MATRIX $\Lambda = (\Lambda^i{}_j)$ IS INVERTIBLE AND WE WILL WRITE THE ENTRIES OF ITS INVERSE AS

$$\Lambda^{-1} = (\Lambda_i{}^j)$$

SO

$$\begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{pmatrix} = \begin{pmatrix} \Lambda_1{}^1 & \cdots & \Lambda_1{}^n \\ \vdots & & \vdots \\ \Lambda_n{}^1 & \cdots & \Lambda_n{}^n \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

$$\hat{e}_i = \Lambda_i{}^j e_j \quad , \quad i = 1, \ldots, n$$

EXERCISE 94: SHOW THAT THE DUAL BASES ARE RELATED BY

$$e^j = \Lambda_i{}^j \hat{e}^i \quad , \quad j = 1, \ldots, n$$

AND

$$\hat{e}^i = \Lambda^i{}_j e^j \quad , \quad i = 1, \ldots, n \quad .$$

EXAMPLE: (U, φ) AND $(\hat{U}, \hat{\varphi})$ CHARTS ON X WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$ AND $\hat{x}^1, \ldots, \hat{x}^n$, RESPECTIVELY. AT ANY POINT IN $U \cap \hat{U}$,

$$\frac{\partial}{\partial x^j} = \frac{\partial \hat{x}^i}{\partial x^j} \frac{\partial}{\partial \hat{x}^i} \qquad \frac{\partial}{\partial \hat{x}^i} = \frac{\partial x^j}{\partial \hat{x}^i} \frac{\partial}{\partial x^j}$$

$$dx^j = \frac{\partial x^j}{\partial \hat{x}^i} d\hat{x}^i \qquad d\hat{x}^i = \frac{\partial \hat{x}^i}{\partial x^j} dx^j$$

CLASSICALLY, THESE ARE CALLED THE "TRANSFORMATION LAWS" FOR CONTRAVARIANT AND COVARIANT VECTORS.

NOW WE TURN TO BILINEAR FORMS ON V :

A MAP

$$A : V \times V \longrightarrow \mathbb{R}$$

SATISFYING

$$A(a_1 v_1 + a_2 v_2 , w) = a_1 A(v_1, w) + a_2 A(v_2, w)$$

AND

$$A(v, a_1 w_1 + a_2 w_2) = a_1 A(v, w_1) + a_2 A(v, w_2)$$

$\forall\ a_1, a_2 \in \mathbb{R}\ \forall\ v, w, v_1, v_2, w_1, w_2 \in V$ IS CALLED A <u>BILINEAR FORM</u> ON V.

THE COLLECTION OF ALL BILINEAR FORMS ON V IS A VECTOR SPACE UNDER POINTWISE ADDITION AND SCALAR MULTIPLICATION :

$$(A+B)(v, w) = A(v, w) + B(v, w)$$

$$(aA)(v, w) = a\,(A(v, w))$$

FOR ANY TWO ELEMENTS α AND β IN THE DUAL V^* OF V WE DEFINE A BILINEAR FORM

$$\alpha \otimes \beta$$

CALLED THE <u>TENSOR PRODUCT</u> OF α AND β BY

$$(\alpha \otimes \beta)(v, w) = \alpha(v)\beta(w).$$

LEMMA : LET $\{e_1, \ldots, e_n\}$ BE A BASIS FOR V AND $\{e^1, \ldots, e^n\}$ THE DUAL BASIS FOR V^*. THEN

$$\{ e^i \otimes e^j \;:\; i, j = 1, \ldots, n \}$$

IS A BASIS FOR THE SPACE OF ALL BILINEAR FORMS ON V.

PROOF : LET A BE A BILINEAR FORM ON V. FOR ANY $v, w \in V$ WRITE $v = v^i e_i$ AND $w = w^j e_j$. THEN

$$\begin{aligned} A(v, w) &= A(v^i e_i, w^j e_j) \\ &= v^i w^j A(e_i, e_j) \\ &= A(e_i, e_j) e^i(v) e^j(w) \\ &= A(e_i, e_j)(e^i \otimes e^j)(v, w) \end{aligned}$$

SO

$$\boxed{\begin{aligned} A &= A(e_i, e_j)\, e^i \otimes e^j \\ &= A_{ij}\, e^i \otimes e^j \end{aligned}}$$

AND THE $e^i \otimes e^j$ SPAN THE SPACE OF BILINEAR FORMS ON V. THEY ARE INDEPENDENT SINCE, IF $A_{ij} e^i \otimes e^j$ IS THE ZERO BILINEAR FORM, THEN FOR ANY a AND b,

$$0 = (A_{ij} e^i \otimes e^j)(e_a, e_b) = A_{ij}\, e^i(e_a) e^j(e_b) = A_{ab}. \qquad \square$$

THE SPACE OF BILINEAR FORMS ON V IS DENOTED

$$V^* \otimes V^*$$

AND WE HAVE JUST SHOWN THAT

$$\dim (V^* \otimes V^*) = n^2 .$$

THE ELEMENTS OF $V^* \otimes V^*$ ARE ALSO CALLED <u>COVARIANT TENSORS OF RANK 2 ON</u> V.

A BILINEAR FORM A ON V IS

<u>SYMMETRIC</u> IF $A(w, v) = A(v, w) \ \forall v, w \in V$

<u>SKEW-SYMMETRIC</u> IF $A(w, v) = -A(v, w) \ \forall v, w \in V$

<u>NONDEGENERATE</u> IF $A(v, w) = 0 \ \forall v \in V \Rightarrow w = 0 \in V$

<u>POSITIVE DEFINITE</u> IF $A(v, v) \geq 0$ AND $A(v, v) = 0 \Leftrightarrow v = 0$

A SKEW-SYMMETRIC BILINEAR FORM ON V IS CALLED A <u>2-FORM</u> ON V. WE MAY GET BACK TO THESE LATER, BUT FOR NOW WE WILL BE INTERESTED IN THE FOLLOWING TYPE OF BILINEAR FORMS :

AN <u>INNER PRODUCT</u> ON V IS A BILINEAR FORM $g \in V^* \otimes V^*$ THAT IS SYMMETRIC AND POSITIVE DEFINITE.

<u>NOTE</u> : g IS NECESSARILY NONDEGENERATE.

OTHER COMMON NOTATIONS :

$$g(v, w) = \langle v, w \rangle = v \cdot w$$

(BUT DO NOT CONFUSE THE LAST TWO WITH THE STANDARD INNER PRODUCT ON $\mathbb{R}^n$, WHICH IS JUST ONE PARTICULAR EXAMPLE)

IF $\{e_1, \ldots, e_n\}$ IS A BASIS FOR V WITH DUAL BASIS $\{e^1, \ldots, e^n\}$ FOR V^*, THEN

$$g = g_{ij}\, e^i \otimes e^j$$

WHERE

$$g_{ij} = g(e_i, e_j)$$

IN ANOTHER BASIS $\{\hat{e}_1, \ldots, \hat{e}_n\}$,

$$g = \hat{g}_{ij}\, \hat{e}^i \otimes \hat{e}^j$$

WHERE

$$\hat{g}_{ij} = g(\hat{e}_i, \hat{e}_j) = g(\Lambda_i{}^k e_k, \Lambda_j{}^l e_l)$$

$$= \Lambda_i{}^k \Lambda_j{}^l g(e_k, e_l)$$

$$\hat{g}_{ij} = \Lambda_i{}^k \Lambda_j{}^l g_{kl}$$

THIS IS WORTH SUMMARIZING : TRANSFORMATION LAW FOR COVARIANT TENSORS OF RANK 2 :

$$\hat{e}_i = \Lambda_i{}^k e_k \quad \Rightarrow \quad \hat{g}_{ij} = \Lambda_i{}^k \Lambda_j{}^\ell g_{k\ell}$$

EXAMPLE : SUPPOSE g_p IS AN INNER PRODUCT ON $T_p(X)$ AND $\{ \frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p \}$ IS A COORDINATE BASIS FOR $T_p(X)$. THE DUAL BASIS FOR $T_p^*(X)$ IS $\{ dx_p^1, \ldots, dx_p^n \}$ AND

$$g_p = g_{ij}(p)\, dx_p^i \otimes dx_p^j$$

WHERE

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x^i}\Big|_p , \frac{\partial}{\partial x^j}\Big|_p \right)$$

IF $\{ \frac{\partial}{\partial \hat{x}^1}|_p, \ldots, \frac{\partial}{\partial \hat{x}^n}|_p \}$ IS ANOTHER COORDINATE BASIS WITH DUAL BASIS $\{ d\hat{x}_p^1, \ldots, d\hat{x}_p^n \}$, THEN

$$g_p = \hat{g}_{ij}(p)\, d\hat{x}_p^i \otimes d\hat{x}_p^j$$

AND

$$\hat{g}_{ij}(p) = \frac{\partial x^k}{\partial \hat{x}^i}(p) \frac{\partial x^\ell}{\partial \hat{x}^j}(p)\, g_{k\ell}(p)$$

THIS IS EASIER TO REMEMBER WITHOUT ALL THE p'S :

$$\hat{g}_{ij} = \frac{\partial x^k}{\partial \hat{x}^i} \frac{\partial x^l}{\partial \hat{x}^j} g_{kl}$$

NOW WE'RE FINALLY READY FOR THE DEFINITION OF A "RIEMANNIAN METRIC" ON A MANIFOLD.

LET X BE A SMOOTH MANIFOLD OF DIMENSION n. A RIEMANNIAN METRIC ON X IS A RULE g WHICH ASSIGNS TO EACH $p \in X$ AN INNER PRODUCT

$$g_p = g(p) = \langle \, , \rangle_p \; : T_p(X) \times T_p(X) \to \mathbb{R}$$

ON THE TANGENT SPACE $T_p(X)$ AND IS SMOOTH IN THE FOLLOWING SENSE : IF (U, φ) IS ANY CHART ON X WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$, THEN THE COMPONENT FUNCTIONS

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x^i}\Big|_p , \frac{\partial}{\partial x^j}\Big|_p \right)$$

ARE SMOOTH ON U. BECAUSE OF THE TRANSFORMATION LAW ABOVE THIS DEFINITION OF "SMOOTHNESS" DOES NOT DEPEND ON THE CHOICE OF COORDINATES.

LOCALLY, IN COORDINATES,

$$g = g_{ij}\, dx^i \otimes dx^j$$

WHERE

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$$

<u>NOTE</u>: IN THE PHYSICS (AND OLD FASHIONED MATHEMATICS) LITERATURE ONE OFTEN SEES THIS ABBREVIATED

$$g_{ij}\, dx^i dx^j$$

AND CALLED A "LINE ELEMENT".

A <u>RIEMANNIAN MANIFOLD</u> IS A PAIR (X, g) CONSISTING OF A SMOOTH MANIFOLD X AND A RIEMANNIAN METRIC g ON X.

> <u>NOTE</u>: USING A DEVICE KNOWN AS A "PARTITION OF UNITY" ONE CAN SHOW THAT A RIEMANNIAN METRIC CAN BE DEFINED ON ANY SMOOTH MANIFOLD. WE WILL CONCENTRATE ON A NUMBER OF CONCRETE, EXPLICIT EXAMPLES.

TO DESCRIBE THESE EXAMPLES AND TO BEGIN DOING SOME GEOMETRY IN THEM WE WILL NEED A FEW MORE TOOLS AND DEFINITIONS.

SUPPOSE (X, g) IS A RIEMANNIAN MANIFOLD, X' IS A SMOOTH MANIFOLD AND

$$F : X' \longrightarrow X$$

IS AN IMMERSION (RECALL THAT THIS MEANS THAT F IS SMOOTH AND ITS DERIVATIVE $F_{*p} : T_p(X') \rightarrow T_{F(p)}(X)$ IS ONE-TO-ONE FOR EACH $p \in X'$).

DEFINE

$$g' = F^* g$$

(THE <u>PULLBACK</u> OF g BY F) AS FOLLOWS: FOR EACH $p \in X'$, $v_p, w_p \in T_p(X')$,

$$g'_p(v_p, w_p) = (F^* g)_p(v_p, w_p) = g_{F(p)}(F_{*p}(v_p), F_{*p}(w_p))$$

<u>EXERCISE</u> 95: SHOW THAT, FOR EACH $p \in X'$, $g'_p = (F^* g)_p$ IS AN INNER PRODUCT ON $T_p(X')$.

TO SHOW THAT $F^* g$ IS SMOOTH WE WORK OUT A FORMULA FOR COMPUTING IT LOCALLY IN COORDINATES.

$$F : X' \to X$$

(U, φ) A CHART ON X WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$.

(V, ψ) A CHART ON X' WITH COORDINATE FUNCTIONS $y^1, \ldots, y^m$

ASSUME WITHOUT LOSS OF GENERALITY THAT $F(V) \subseteq U$. ON V,

$$(\varphi \circ F \circ \psi^{-1})(y^1, \ldots, y^m) = (x^1, \ldots, x^n)$$

$$= (F^1(y^1, \ldots, y^m), \ldots, F^n(y^1, \ldots, y^m))$$

AND, ON U,

$$\boxed{g = g_{ij}(x^1, \ldots, x^n)\, dx^i \otimes dx^j}$$

WE CLAIM THAT, ON V, F^*g CAN BE OBTAINED FROM g BY SIMPLY SUBSTITUTING $x^k = F^k(y^1, \ldots, y^m)$ EVERYWHERE, I.E., ON V,

$$\boxed{\begin{gathered} F^*g = g_{ij}(F^1(y^1, \ldots, y^m), \ldots, F^n(y^1, \ldots, y^m))\, dF^i \otimes dF^j \\ dF^i = \frac{\partial F^i}{\partial y^k}\, dy^k \end{gathered}}$$

(FROM WHICH SMOOTHNESS IS CLEAR SINCE THE RESULTING COMPONENTS ARE SMOOTH)

TO SEE WHY THIS IS TRUE WE JUST WRITE OUT THE RIGHT-HAND SIDE OF

$$(F^*g)_p(v, w) = g_{F(p)}(F_{*p}(v), F_{*p}(w))$$

IN LOCAL COORDINATES ON X:

LEAVING $p \in V$ ARBITRARY,

$$g_{F(p)}(F_{*p}(v), F_{*p}(w)) = g_{ij}(F^1(y^1,\dots,y^m), \dots F^n(y^1,\dots,y^m))(F_{*p}(v))^i(F_{*p}(w))^j$$

BUT

$$\begin{aligned} F_{*p}(v) &= (F_{*p}(v))(x^i)\frac{\partial}{\partial x^i} \\ &= v(x^i \circ F)\frac{\partial}{\partial x^i} \\ &= v(F^i)\frac{\partial}{\partial x^i} \end{aligned}$$

SO

$$\begin{aligned} (F_{*p}(v))^i &= v(F^i) = (v^j \frac{\partial}{\partial y^j})(F^i) \\ &= \frac{\partial F^i}{\partial y^j} v^j \\ &= (\frac{\partial F^i}{\partial y^j} dy^j)(v) \\ &= dF^i(v) \end{aligned}$$

SIMILARLY FOR $(F_{*p}(w))^j$ SO

$$\begin{aligned} (F_{*p}(v))^i(F_{*p}(w))^j &= (dF^i(v))(dF^j(w)) \\ &= (dF^i \otimes dF^j)(v,w) \end{aligned}$$

THUS,

$$(F^*g)_p(v,w) = g_{ij}(F^1(y^1,\dots,y^m),\dots, F^n(y^1,\dots,y^m))dF^i \otimes dF^j(v,w)$$

FOR ALL $v, w \in T_p(X')$ AS REQUIRED.

YOU CAN PULL BACK A RIEMANNIAN METRIC BY ANY IMMERSION. THE MOST COMMON CHOICE ARISES AS FOLLOWS:

IF (X, g) IS A RIEMANNIAN MANIFOLD AND X' IS A SUBMANIFOLD OF X, THEN THE INCLUSION MAP

$$\iota : X' \hookrightarrow X$$

IS AN IMMERSION. THE RIEMANNIAN METRIC

$$g' = \iota^* g$$

ON X' IS CALLED THE <u>RESTRICTION</u> OF g TO X' (RECALL THAT WE USE ι_* TO IDENTIFY TANGENT SPACES TO X' WITH SUBSPACES OF TANGENT SPACES TO X).

WE WILL SEE MANY EXAMPLES OF THIS SORT SHORTLY.

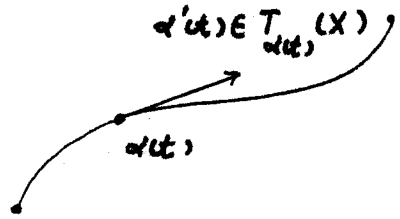
LET (M, g) AND (N, h) BE RIEMANNIAN MANIFOLDS. A DIFFEOMORPHISM $F : M \to N$ IS SAID TO BE AN <u>ISOMETRY</u> IF $F^* h = g$ AND IN THIS CASE WE SAY THAT (M, g) AND (N, h) ARE <u>ISOMETRIC</u>.

WE COULD BEGIN OUR LIST OF EXAMPLES AT THIS POINT, BUT IN ORDER TO HAVE SOMETHING INTERESTING TO COMPUTE IN EACH ONE I WILL INTRODUCE ONE MORE CONCEPT.

IN A RIEMANNIAN MANIFOLD ONE CAN COMPUTE "INNER PRODUCTS" OF TANGENT VECTORS ($g_p(v_p, w_p)$) AND SO ONE CAN DEFINE THE g-<u>LENGTH</u> OF A TANGENT VECTOR BY

$$\|v_p\|_g = \sqrt{g_p(v_p, v_p)}$$

IN PARTICULAR, IF $\alpha : [t_1, t_2] \to X$ IS A SMOOTH CURVE IN (X, g), ONE CAN COMPUTE THE LENGTH OF ITS VELOCITY VECTOR AT EACH t.



$$\|\alpha'(t)\|_g = \sqrt{g_{\alpha(t)}(\alpha'(t), \alpha'(t))}$$

AND FROM THIS ONE CAN DEFINE THE g-<u>LENGTH</u> OF α BY

$$L_g(\alpha) = \int_{t_1}^{t_2} \|\alpha'(t)\|_g \, dt$$

$$= \int_{t_1}^{t_2} \sqrt{g_{\alpha(t)}(\alpha'(t), \alpha'(t))} \, dt$$

NOTICE THAT $L_g(\alpha)$ CAN BE DEFINED FOR A CURVE α THAT IS ONLY <u>PIECEWISE</u> <u>SMOOTH</u> BY ADDING UP THE LENGTHS OF THE SMOOTH PIECES.

EXAMPLES OF RIEMANNIAN MANIFOLDS :

1. THE USUAL RIEMANNIAN METRIC ON $\mathbb{R}^n$

SINCE WE HAVE A GLOBAL COORDINATE SYSTEM $x^1, \dots, x^n$ WE NEED ONLY DEFINE THE COMPONENTS $g_{ij}(x^1, \dots, x^n)$ AND THIS WILL UNIQUELY DETERMINE THE METRIC

$$g = g_{ij}(x^1, \dots, x^n)\, dx^i \otimes dx^j$$

GLOBALLY. WE TAKE THE CONSTANT FUNCTIONS

$$g_{ij}(x^1, \dots, x^n) = \delta_{ij}$$

SO THAT

$$g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$$

AND SO, AT ANY $p \in \mathbb{R}^n$,

$$\begin{aligned} g_p(v_p, w_p) &= g_p\left(v^i \frac{\partial}{\partial x^i}\Big|_p , w^j \frac{\partial}{\partial x^j}\Big|_p\right) \\ &= g_{ij}(p)\, v^i w^j \\ &= \delta_{ij}\, v^i w^j \\ &= v^1 w^1 + \dots + v^n w^n \end{aligned}$$

IN ANOTHER CHART FOR $\mathbb{R}^n$ THE METRIC WILL LOOK QUITE DIFFERENT, OF COURSE. IN FACT, WE HAVE SEEN THAT THE TRANSFORMATION

LAW FOR THE METRIC COMPONENTS IS

$$\hat{g}_{ij} = \frac{\partial x^k}{\partial \hat{x}^i} \frac{\partial x^l}{\partial \hat{x}^j} g_{kl}$$

FOR EXAMPLE, ON $\mathbb{R}^3$ WE HAVE A SPHERICAL COORDINATE CHART $\hat{x}^1 = \rho$, $\hat{x}^2 = \phi$, $\hat{x}^3 = \theta$, RELATED TO x^1, x^2, x^3 BY

$$x^1 = \rho \sin\phi \sin\theta$$
$$x^2 = \rho \sin\phi \cos\theta$$
$$x^3 = \rho \cos\phi$$

SO THE METRIC COMPONENTS IN THIS CHART ARE GIVEN BY

$$\begin{aligned} \hat{g}_{ij} &= \frac{\partial x^k}{\partial \hat{x}^i} \frac{\partial x^l}{\partial \hat{x}^j} \delta_{kl} \\ &= \frac{\partial x^1}{\partial \hat{x}^i} \frac{\partial x^1}{\partial \hat{x}^j} + \frac{\partial x^2}{\partial \hat{x}^i} \frac{\partial x^2}{\partial \hat{x}^j} + \frac{\partial x^3}{\partial \hat{x}^i} \frac{\partial x^3}{\partial \hat{x}^j} \end{aligned}$$

E.G., FOR $i = j = 1$

$$\begin{aligned} \hat{g}_{11} = \hat{g}_{\rho\rho} &= \left(\frac{\partial x^1}{\partial \rho}\right)^2 + \left(\frac{\partial x^2}{\partial \rho}\right)^2 + \left(\frac{\partial x^3}{\partial \rho}\right)^2 \\ &= (\sin\phi \sin\theta)^2 + (\sin\phi\cos\theta)^2 + (\cos\phi)^2 \\ &= 1 \end{aligned}$$

<u>EXERCISE</u> 96: COMPUTE THE REMAINING $\hat{g}_{ij}$ AND CONCLUDE THAT, IN SPHERICAL COORDINATES ON $\mathbb{R}^3$,

$$g = d\rho \otimes d\rho + \rho^2 (d\phi \otimes d\phi + \sin^2\phi \, d\theta \otimes d\theta).$$

2. THE USUAL RIEMANNIAN METRIC ON S^n

THIS IS THE RESTRICTION OF THE USUAL RIEMANNIAN METRIC ON $\mathbb{R}^{n+1}$ TO S^n, I.E., IF

$$\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$$

IS THE INCLUSION MAP, THEN

$$g_{S^n} = \iota^* g_{\mathbb{R}^{n+1}}$$

FOR $p \in S^n$ AND $v_p, w_p \in T_p(S^n)$,

$$\begin{aligned} g_{S^n}(p)(v_p, w_p) &= (\iota^* g_{\mathbb{R}^{n+1}})(p)(v_p, w_p) \\ &= g_{\mathbb{R}^{n+1}}(\iota(p))(\iota_{*p}(v_p), \iota_{*p}(w_p)) \\ &= g_{\mathbb{R}^{n+1}}(p)(v_p, w_p) \end{aligned}$$

SINCE WE USE ι_{*p} TO IDENTIFY $T_p(S^n)$ WITH A SUBSPACE OF $T_p(\mathbb{R}^{n+1})$.

THIS JUST SAYS THAT g_{S^n} LOOKS AT TANGENT VECTORS TO S^n AS IF THEY WERE TANGENT VECTORS TO $\mathbb{R}^{n+1}$ AND COMPUTES $g_{\mathbb{R}^{n+1}}$ OF THEM.

TO WRITE THIS OUT IN COORDINATES ON S^n (AS OPPOSED TO $x^1, \ldots, x^{n+1}$) WE JUST WRITE THE INCLUSION MAP IN THESE COORDINATES AND COMPUTE THE PULLBACK, E.G., WITH $y^1 = \phi$, $y^2 = \theta$ ON S^2,

$$\iota(y^1, y^2) = (x^1, x^2, x^3)$$
$$= (\sin\phi \sin\theta, \sin\phi\cos\theta, \cos\phi)$$

AND $g_{ij}(x^1, x^2, x^3) = \delta_{ij}$, THE PULLBACK IS COMPUTED AS FOLLOWS (I WILL USE THE EASILY VERIFIED DISTRIBUTIVITY OF THE TENSOR PRODUCT):

$$g_{ij}(\sin\phi\sin\theta, \sin\phi\cos\theta, \cos\phi)\, dx^i \otimes dx^j = \delta_{ij}\, dx^i \otimes dx^j =$$

$$dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 = d(\sin\phi\sin\theta) \otimes d(\sin\phi\sin\theta) +$$
$$d(\sin\phi\cos\theta) \otimes d(\sin\phi\cos\theta +$$
$$d(\cos\phi) \otimes d(\cos\phi) =$$

$$(\cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta) \otimes (\cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta)$$

$$+ (\cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta) \otimes (\cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta)$$

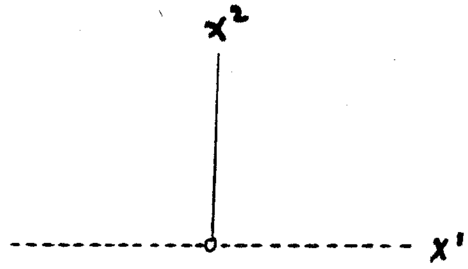
$$+ (-\sin\phi\, d\phi) \otimes (-\sin\phi\, d\phi)$$

$$= \cos^2\phi\sin^2\theta\, d\phi \otimes d\phi + \sin^2\phi\cos^2\theta\, d\theta \otimes d\theta +$$
$$\cos^2\phi\cos^2\theta\, d\phi \otimes d\phi + \sin^2\phi\sin^2\theta\, d\theta \otimes d\theta +$$
$$\sin^2\phi\, d\phi \otimes d\phi$$

$$= d\phi \otimes d\phi + \sin^2\phi\, d\theta \otimes d\theta$$

3. POINCARÉ UPPER HALF-PLANE

$$X = \{ (x^1, x^2) \in \mathbb{R}^2 : x^2 > 0 \}$$



WE USE THE GLOBAL COORDINATE SYSTEM x^1, x^2 TO SPECIFY THE METRIC

$$g = g_{ij}(x^1, x^2)\, dx^i \otimes dx^j$$

BY CHOOSING

$$g_{11}(x^1,x^2) = g_{22}(x^1,x^2) = \frac{1}{(x^2)^2}$$

$$g_{12}(x^1,x^2) = g_{21}(x^1,x^2) = 0.$$

THUS,

$$g = \frac{1}{(x^2)^2}\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$$

<u>EXERCISE</u> 97: SHOW THAT THIS IS A RIEMANNIAN METRIC ON X.

<u>NOTE</u>: IF $p = (p^1, p^2)$ AND $q = (q^1, q^2)$ ARE IN X AND $v_p \in T_p(X)$, $v_q \in T_q(X)$ HAVE THE SAME COMPONENTS RELATIVE TO THE STANDARD COORDINATE BASIS, SAY, $v_p = 1 \frac{\partial}{\partial x^1}\Big|_p + 1 \frac{\partial}{\partial x^2}\Big|_p$ AND $v_q = 1 \frac{\partial}{\partial x^1}\Big|_q + 1 \frac{\partial}{\partial x^2}\Big|_q$, THEN

$$\| v_p \|_g^2 = g_p(v_p, v_p) = \frac{1}{(p^2)^2}(1^2 + 1^2) = \frac{2}{(p^2)^2}$$

AND

$$\| v_q \|_g^2 = g_q(v_q, v_q) = \frac{1}{(q^2)^2}(1^2 + 1^2) = \frac{2}{(q^2)^2}$$

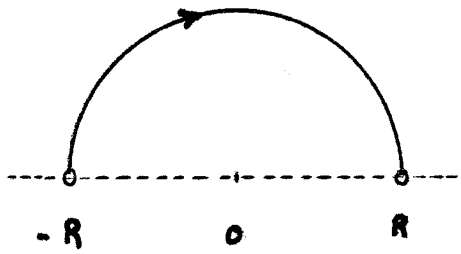
GENERALLY NOT EQUAL

JUST FOR FUN LET'S COMPUTE THE LENGTH OF A CURVE IN THE POINCARÉ UPPER HALF-PLANE. THE SMOOTH CURVE

$$\begin{aligned} \alpha(t) &= (x^1(t), x^2(t)) \\ &= (-R\cos t,\ R\sin t) \end{aligned}$$

$$0 < t < \pi$$

IS THE UPPER SEMICIRCLE OF RADIUS R ABOUT THE ORIGIN.



TO FIND ITS LENGTH WE COMPUTE $\displaystyle\int_{0+\varepsilon}^{\pi-\varepsilon} \sqrt{g_{\alpha(t)}(\alpha'(t), \alpha'(t))}\, dt$ AND LET $\varepsilon \to 0^+$.

$$\alpha'(t) = (R\sin t)\frac{\partial}{\partial x^1}\Big|_{\alpha(t)} + (R\cos t)\frac{\partial}{\partial x^2}\Big|_{\alpha(t)}$$

SO

$$\begin{aligned} g_{\alpha(t)}(\alpha'(t), \alpha'(t)) &= \frac{1}{(R\sin t)^2}\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)(\alpha'(t), \alpha'(t)) \\ &= \frac{1}{R^2\sin^2 t}\,(R^2\sin^2 t + R^2\cos^2 t) \\ &= \frac{1}{\sin^2 t} \end{aligned}$$

THUS,

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\pi-\varepsilon} \frac{1}{\sin t}\, dt = \lim_{\varepsilon \to 0^+} \ln|\csc t - \cot t|\Big|_{\varepsilon}^{\pi-\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \left[\ln|\csc(\pi-\varepsilon) - \cot(\pi-\varepsilon)| - \ln|\csc\varepsilon - \cot\varepsilon| \right]$$

$$= \lim_{\varepsilon \to 0^+} \left[\ln|\csc\varepsilon + \cot\varepsilon| - \ln|\csc\varepsilon - \cot\varepsilon| \right]$$

$$= \lim_{\varepsilon \to 0^+} \ln \left| \frac{\csc\varepsilon + \cot\varepsilon}{\csc\varepsilon - \cot} \right|$$

$$= \lim_{\varepsilon \to 0^+} \ln \left| \frac{1+\cos\varepsilon}{1-\cos\varepsilon} \right| = \infty$$

THE SEMICIRCLE IS RATHER LONG IN (X, g). THIS IS RATHER AMUSING SINCE ONE CAN SHOW THAT THESE SEMICIRCLES ARE ACTUALLY THE "GEODESICS" ("STRAIGHT LINES") IN THE POINCARÉ UPPER HALF-PLANE.

4. THE HYPERBOLIC DISC

HERE WE WILL PRESENT WHAT IS ESSENTALLY JUST ANOTHER WAY OF LOOKING AT (I.E., AN ISOMETRIC COPY OF) THE POINCARÉ UPPER HALF-PLANE. LET

$$B_1^2(0) = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \}$$

BE THE OPEN DISC OF RADIUS 1 ABOUT THE ORIGIN IN $\mathbb{R}^2$ WITH ITS USUAL DIFFERENTIABLE STRUCTURE, BUT WITH A RIEMANNIAN

METRIC DEFINED BY

$$g = \frac{4}{(1-x^2-y^2)^2}(dx \otimes dx + dy \otimes dy)$$

NOTE: THIS CLEARLY IS A RIEMANNIAN METRIC ON $B_1^2(0)$. NOTE ALSO THAT $\frac{4}{(1-x^2-y^2)^2} \to \infty$ AS (x,y) APPROACHES THE BOUNDARY OF $B_1^2(0)$ JUST AS, IN THE POINCARÉ UPPER HALF-PLANE, $\frac{1}{(x^2)^2} \to \infty$ AS (x^1, x^2) APPROACHES THE BOUNDARY.

TO EXHIBIT AN ISOMETRY BETWEEN THESE TWO RIEMANNIAN MANIFOLDS WE BORROW SOMETHING FROM COMPLEX ANALYSIS: DEFINE

$$F : \{z \in \mathbb{C} : |z| < 1\} \to \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

BY

$$F(z) = \frac{i+z}{1+iz}$$

(FRACTIONAL LINEAR TRANSFORMATION)

EXERCISE 98: WRITE F OUT IN TERMS OF REAL AND IMAGINARY PARTS TO SHOW THAT IT IS SMOOTH AS A MAP FROM THE OPEN UNIT DISC TO THE OPEN UPPER HALF-PLANE. FIND F^{-1} AND DO THE SAME FOR IT. THUS, F IS A DIFFEOMORPHISM. NOW, FROM $(x^1, x^2) = F(x,y)$ PULL BACK THE METRIC ON THE POINCARÉ UPPER HALF-PLANE AND SHOW THAT YOU GET THE METRIC g ON $B_1^2(0)$ SHOWN ABOVE. THUS, F IS AN ISOMETRY.

5. n-DIMENSIONAL HYPERBOLIC SPACE

BY DIRECT ANALOGY WITH THE HYPERBOLIC DISC,

$$B_1^n(0) = \{ x \in \mathbb{R}^n : \|x\| < 1 \}$$

$$g = \frac{4}{(1 - \|x\|^2)^2} (dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)$$

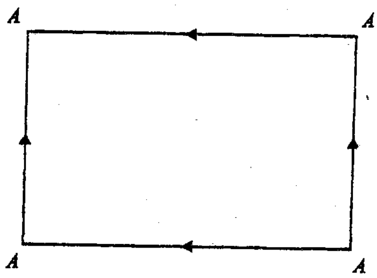
6. THE FLAT TORUS

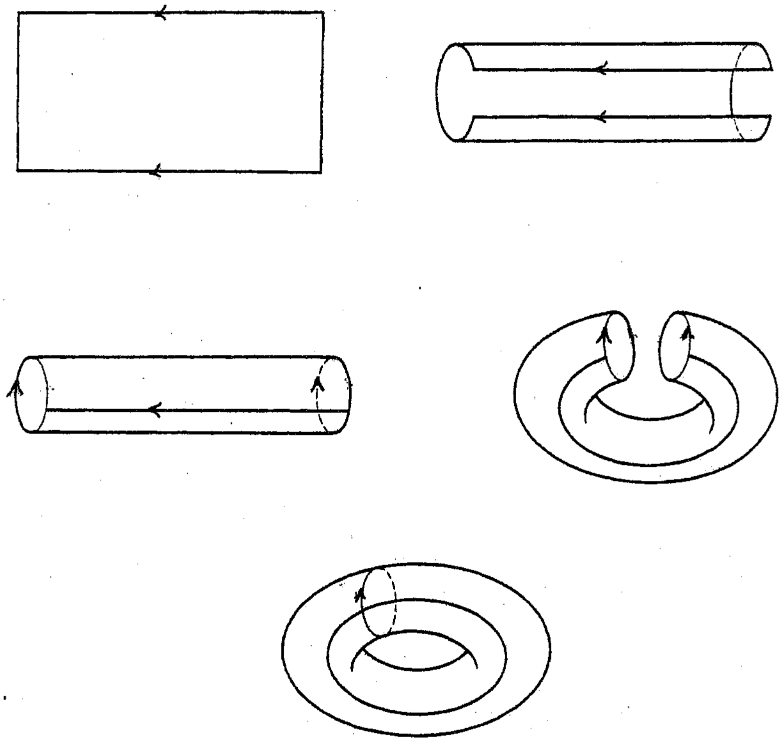
HERE'S A VERY HARD QUESTION THAT GEOMETERS SPEND A GREAT DEAL OF TIME THINKING ABOUT :

> GIVEN A MANIFOLD, WHAT "KIND" OF RIEMANNIAN METRICS CAN EXIST ON IT ?

"KIND" USUALLY HAS SOMETHING TO DO WITH "CURVATURE" WHICH WE HAVE NOT DISCUSSED. FOR THE PRESENT I JUST WANT TO SHOW THAT THE TORUS $S^1 \times S^1$ HAS DEFINED ON IT A RIEMANNIAN METRIC WHICH, LOCALLY ON A NEIGHBORHOOD OF ANY POINT, LOOKS JUST LIKE THE METRIC $dx^1 \otimes dx^1 + dx^2 \otimes dx^2$ OF THE PLANE.

THE TORUS CAN BE OBTAINED BY IDENTIFYING THE EDGES OF A RECTANGLE IN THE FOLLOWING WAY :





NOW HERE'S WHAT WE WANT TO DO : THE RECTANGLE WE START WITH LIVES IN THE PLANE $\mathbb{R}^2$ AND SO HAS IT'S USUAL "FLAT" METRIC $dx \otimes dx + dy \otimes dy$ (WE WON'T DEFINE "FLAT" PRECISELY FOR A LITTLE WHILE, BUT YOU KNOW WHAT I HAVE IN MIND). WE WANT TO MOVE THIS METRIC DOWN TO THE TORUS TO GET A "FLAT" METRIC THERE ALSO.

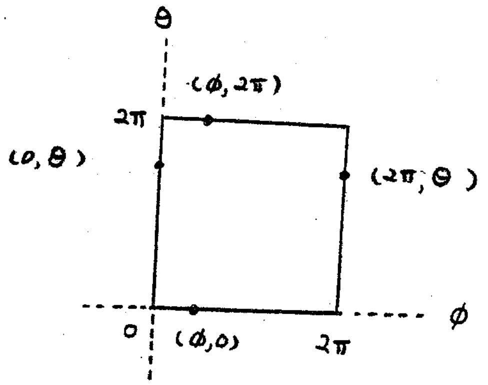
THIS EXAMPLE IS QUITE IMPORTANT AND INTERESTING (E.G., EVEN THOUGH EVERY COMPACT SURFACE CAN BE GOTTEN THIS WAY BY IDENTIFYING EDGES ON SOME POLYGON, NO OTHER SURFACE HAS SUCH A "FLAT" METRIC - THE REASON IS TOPOLOGICAL).

"IDENTIFYING" EDGES, OF COURSE, MEANS FORMING A QUOTIENT SPACE. RATHER THAN DOING THIS ABSTRACTLY, HOWEVER, WE'LL WRITE DOWN A CONCRETE MAPPING THAT DOES THE IDENTIFYING.

BEGIN WITH A SMOOTH MAP $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ GIVEN BY

$$P(\phi,\theta) = (x^1, x^2, x^3, x^4)$$
$$= (\cos(\phi+\theta), \sin(\phi+\theta), \cos(\phi-\theta), \sin(\phi-\theta))$$

NOTICE THAT, RESTRICTED TO $[0,2\pi] \times [0,2\pi] \subseteq \mathbb{R}^2$,



P IS ONE-TO-ONE ON $(0,2\pi) \times (0,2\pi)$, BUT TAKES THE SAME VALUES AT $(\phi,0)$ AND $(\phi,2\pi)$ FOR $0 \le \phi \le 2\pi$ AND ALSO AT $(0,\theta)$ AND $(2\pi,\theta)$ FOR $0 \le \theta \le 2\pi$ (IN THE IMAGE OF P THESE POINTS WILL BE "IDENTIFIED").

SINCE P REPEATS ON SQUARES OF SIDE LENGTH 2π, THE IMAGE OF P IS THE SAME AS THE IMAGE OF $P|[0,2\pi] \times [0,2\pi]$. NOTICE THAT THIS IMAGE IS HOMEOMORPHIC TO THE SUBSPACE $S^1 \times S^1$ OF $\mathbb{R}^4$ VIA A RATHER TRIVIAL MAP:

$$(\cos(\phi+\theta), \sin(\phi+\theta), \cos(\phi-\theta), \sin(\phi-\theta)) \longrightarrow$$

$$((\cos(\phi+\theta), \sin(\phi+\theta)), (\cos(\phi-\theta), \sin(\phi-\theta))) =$$

$$(e^{i(\phi+\theta)}, e^{i(\phi-\theta)})$$

THUS, WE'LL IDENTIFY THE IMAGE OF P WITH THE TORUS TOPOLOGICALLY.

> <u>NOTE</u>: THE IMAGE OF P REALLY DOES HAVE THE QUOTIENT TOPOLOGY DETERMINED BY $P|[0,2\pi] \times [0,2\pi]$ BECAUSE CLOSED MAPS ARE ALWAYS QUOTIENT MAPS AND $[0,2\pi] \times [0,2\pi]$ IS COMPACT.

NOW TURN TO THE DIFFERENIABLE STRUCTURES. FOR THE TORUS $S^1 \times S^1$ WE HAVE THE PRODUCT DIFFERENTIABLE STRUCTURE (IF (U_1, φ_1) AND (U_2, φ_2) ARE CHARTS ON S^1, THEN WE OBTAIN A CHART $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ ON $S^1 \times S^1$ BY DEFINING $(\varphi_1 \times \varphi_2)(u_1, u_2) = (\varphi_1(u_1), \varphi_2(u_2)) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$).

FOR THE IMAGE OF P WE WILL SHOW FIRST THAT $P : \mathbb{R}^2 \to \mathbb{R}^4$ IS AN IMMERSION. THIS IS EASY BECAUSE THE JACOBIAN IS

$$\begin{pmatrix} \frac{\partial x^1}{\partial \phi} & \frac{\partial x^1}{\partial \theta} \\ \frac{\partial x^2}{\partial \phi} & \frac{\partial x^2}{\partial \theta} \\ \frac{\partial x^3}{\partial \phi} & \frac{\partial x^3}{\partial \theta} \\ \frac{\partial x^4}{\partial \phi} & \frac{\partial x^4}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -\sin(\phi+\theta) & -\sin(\phi+\theta) \\ \cos(\phi+\theta) & \cos(\phi+\theta) \\ -\sin(\phi-\theta) & \sin(\phi-\theta) \\ \cos(\phi-\theta) & -\cos(\phi-\theta) \end{pmatrix}$$

AND THERE ARE FOUR 2×2 SUBMATRICES WITH DETERMINANTS THAT ARE NOT IDENTICALLY ZERO. THEIR DETERMINANTS ARE

$$-2\sin(\phi+\theta)\sin(\phi-\theta)$$

$$-2\cos(\phi+\theta)\cos(\phi-\theta)$$

$$2\sin(\phi+\theta)\cos(\phi-\theta)$$

$$2\cos(\phi+\theta)\sin(\phi-\theta)$$

AND AT LEAST ONE OF THESE IS NONZERO <u>FOR EACH</u> (ϕ, θ) <u>IN</u> $\underline{\mathbb{R}}^2$.

THUS, $P_{*(\phi,\theta)}$ HAS RANK 2 AT EACH POINT. SINCE $\mathbb{R}^2$ HAS DIMENSION 2, THE DERIVATIVE MUST BE INJECTIVE AT EACH POINT SO P IS AN IMMERSION.

NOW, P IS ONE-TO-ONE (AND THEREFORE AN EMBEDDING) ON $(0,2\pi)\times(0,2\pi)$ AND THIS GIVES A CHART ON THE PIECE $P((0,2\pi)\times(0,2\pi))$ OF THE IMAGE OF P. IN THE USUAL WAY, ALTERING THE DOMAIN (E.G., TO $(-\pi,\pi)\times(-\pi,\pi)$) GIVES MORE CHARTS AND WE CAN COVER THE IMAGE OF P WITH THESE.

IT'S NOT HARD TO SHOW THAT, WITH THESE DIFFERENTIABLE STRUCTURES, THE HOMEOMORPHISM DESCRIBED ABOVE FROM THE IMAGE OF P TO THE TORUS IS A DIFFEOMORPHISM SO WE CAN FROM THIS POINT ON WRITE $S^1\times S^1$ AND MEAN THE IMAGE OF THE MAP $P : \mathbb{R}^2 \to \mathbb{R}^4$.

NOW HERE'S THE POINT OF LOOKING AT THE TORUS IN THIS WAY:

FORGET THE USUAL RIEMANNIAN METRIC ON $S^1 \times S^1$. WE'LL DEFINE A NEW ONE BY PUSHING THE STANDARD METRIC OF $\mathbb{R}^2$ TO THE IMAGE OF P VIA THE DERIVATIVE OF P. MORE PRECISELY :

FOR EACH $p \in S^1 \times S^1$ $(= P(\mathbb{R}^2))$ AND ANY $v_p, w_p \in T_p(S^1 \times S^1)$ DEFINE $g_p(v_p, w_p)$ AS FOLLOWS : $\exists !\ (\phi_0, \theta_0) \in [0, 2\pi) \times [0, 2\pi)$ SUCH THAT $P(\phi_0, \theta_0) = p$. $P_{*(\phi_0,\theta_0)}$ HAS RANK 2 AND SO IS AN ISOMORPHISM OF $T_{(\phi_0,\theta_0)}(\mathbb{R}^2)$ ONTO $T_p(S^1 \times S^1)$. SELECT THE UNIQUE $V_{(\phi_0,\theta_0)}$ AND $W_{(\phi_0,\theta_0)}$ IN $T_{(\phi_0,\theta_0)}(\mathbb{R}^2)$ WITH $P_{*(\phi_0,\theta_0)}(V_{(\phi_0,\theta_0)}) = v_p$ AND $P_{*(\phi_0,\theta_0)}(W_{(\phi_0,\theta_0)}) = w_p$ AND SET

$$g_p(v_p, w_p) = \langle V_{(\phi_0,\theta_0)}, W_{(\phi_0,\theta_0)} \rangle_{\mathbb{R}^2}$$

MORE BRIEFLY,

$$g_p(v_p, w_p) = \langle (P^{-1})_{*p}(v_p), (P^{-1})_{*p}(w_p) \rangle$$

(WHERE P^{-1} MEANS THE LOCAL INVERSE OF P FROM A NEIGHBORHOOD OF p IN $S^1 \times S^1$ TO A NEIGHBORHOOD OF (ϕ_0, θ_0) IN $\mathbb{R}^2$).

THUS, WE HAVE JUST TRANSPLANTED THE USUAL $\mathbb{R}^2$ RIEMANNIAN METRIC TO $S^1 \times S^1$ BY P.

LOCALLY, NEAR ANY POINT, THERE ARE COORDINATES $(x^1 = \phi \circ P^{-1}$ AND $x^2 = \theta \circ P^{-1})$ RELATIVE TO WHICH

$$g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 .$$

$(S^1 \times S^1, g)$ IS CALLED (FOR REASONS WE WILL SEE) THE <u>FLAT</u> <u>TORUS</u>.

PREVIEWS OF COMING ATTRACTIONS

THE EXAMPLE OF THE "FLAT TORUS" POSES A NUMBER OF INTERESTING QUESTIONS.

1. WHY IS IT CALLED "FLAT"? IT DOESN'T LOOK FLAT! PRESUMABLY THE ANSWER SHOULD HAVE SOMETHING TO DO WITH "CURVATURE" SINCE ONE WOULD THINK THAT'S JUST WHAT A "FLAT" THING DOESN'T HAVE.

2. WHY CAN'T WE CONSTRUCT A "FLAT SPHERE", OR A "FLAT 2-HOLED TORUS"? WHAT DOES THIS "CURVATURE" THING HAVE TO DO WITH TOPOLOGY?

3. WHAT EXACTLY IS THIS "CURVATURE" ANYWAY? CAN YOU MEASURE IT? COMPUTE IT? FOR THAT MATTER, CAN YOU DEFINE IT?

ALL OF THESE ARE QUESTIONS IN DIFFERENTIAL GEOMETRY AND NONE OF THEM CAN BE ANSWERED UNTIL YOU ANSWER THE LAST ONE. WE WILL CONCLUDE BY TRYING TO INDICATE WHERE THE ANSWER TO "HOW DO YOU DEFINE CURVATURE?" COMES FROM. ACTUALLY, THE ANSWER COMES FROM THINKING ABOUT AN APPARENTLY SIMPLER QUESTION:

> GIVEN AN n-DIMENSIONAL RIEMANNIAN MANIFOLD (M, g) AND A CHART (U, φ) IN WHICH THE METRIC COMPONENTS ARE g_{ij}, WHEN CAN ONE PERFORM A COORDINATE TRANSFORMATION FOR WHICH THE NEW METRIC COMPONENTS ARE δ_{ij} (I.E., WHEN IS g, DESPITE WHAT IT LOOKS LIKE IN THE CHART (U, φ), "REALLY" FLAT)?

M = SMOOTH n-MANIFOLD WITH RIEMANNIAN METRIC g.

(U, φ) = CHART WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$

(g_{ij}) = METRIC COMPONENTS IN (U, φ), I.E., $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

$(g^{ij}) = (g_{ij})^{-1}$

SUPPOSE THERE IS ANOTHER CHART (V, ψ), COORDINATE FUNCTIONS $y^1, \ldots, y^n$, FOR WHICH

$$g\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = \delta_{ab} = \begin{cases} 1, & a = b = 1, \ldots, n \\ 0, & a \neq b \end{cases}$$

ON V AND WITH $U \cap V \neq \phi$.

TRANSFORMATION EQUATIONS:

$$\psi \circ \varphi^{-1} : \begin{cases} y^1 = y^1(x^1, \ldots, x^n) \\ \vdots \\ y^n = y^n(x^1, \ldots, x^n) \end{cases}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a} \qquad \frac{\partial}{\partial x^j} = \frac{\partial y^b}{\partial x^j} \frac{\partial}{\partial y^b}$$

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = g\left(\frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}, \frac{\partial y^b}{\partial x^j} \frac{\partial}{\partial y^b}\right)$$

$$g_{ij} = \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} g\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right)$$

$$g_{ij} = \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j} \delta_{ab}$$

(1) $$g_{ij} = \sum_{a=1}^{n} \frac{\partial y^a}{\partial x^i} \frac{\partial y^a}{\partial x^j}, \quad i, j = 1, \ldots, n$$

EQUATIONS (1) ARE EQUIVALENT TO THE MATRIX EQUATION

$$(g_{ij}) = \left(\frac{\partial y^a}{\partial x^i}\right)^T \left(\frac{\partial y^a}{\partial x^i}\right)$$

$$\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$$

MOMENTARILY SIMPLIFY THE NOTATION TO

$$G = J^T J .$$

THEN

$$G^{-1} = (J^T J)^{-1} = J^{-1} (J^T)^{-1}$$

OR

$$J G^{-1} J^T = Id$$

NOW BACK TO ENTRIES

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix} \begin{pmatrix} g^{11} & \cdots & g^{1n} \\ \vdots & & \vdots \\ g^{n1} & \cdots & g^{nn} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^n} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

$$(2) \qquad \frac{\partial y^a}{\partial x^i} \, g^{ij} \, \frac{\partial y^b}{\partial x^j} = \delta^{ab} \qquad , a, b = 1, \dots, n$$

NOW, DIFFERENTIATE (1) WITH RESPECT TO x^k TO GET

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_{a=1}^{n} \frac{\partial}{\partial x^k}\left(\frac{\partial y^a}{\partial x^i}\frac{\partial y^a}{\partial x^j} \right) = \sum_{a=1}^{n}\left(\frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^k \partial x^j} + \frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^k \partial x^i} \right)$$

$$= \sum_{a=1}^{n}\left(\frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^i \partial x^k} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} \right)$$

<u>EXERCISE</u> 99: WRITE DOWN SIMILAR EXPRESSIONS FOR $\dfrac{\partial g_{ik}}{\partial x^j}$ AND $\dfrac{\partial g_{jk}}{\partial x^i}$ AND COMBINE THEM TO GET

$$(3) \qquad \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) = \sum_{a=1}^{n} \frac{\partial^2 y^a}{\partial x^j \partial x^k}\frac{\partial y^a}{\partial x^i}$$

ALGEBRAIC TRICK :

FIX SOME INDEX $b = 1, \dots, n$. MULTIPLY ON BOTH SIDES OF (3) BY

$$g^{i\beta}\frac{\partial y^b}{\partial x^\beta} \qquad (\text{SUMMED OVER } \beta = 1, \dots, n)$$

AND SUM OVER i AS REQUIRED BY THE SUMMATION CONVENTION.

$$\frac{1}{2} g^{i\beta}\frac{\partial y^b}{\partial x^\beta}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) = \sum_{a=1}^{n} g^{i\beta}\frac{\partial y^b}{\partial x^\beta}\frac{\partial^2 y^a}{\partial x^j \partial x^k}\frac{\partial y^a}{\partial x^i}$$

$$\frac{\partial y^a}{\partial x^i} g^{i\beta}\frac{\partial y^b}{\partial x^\beta} = \delta^{ab}$$

BY (2)

$$= \sum_{a=1}^{n} \delta^{ab}\frac{\partial^2 y^a}{\partial x^j \partial x^k}$$

$$(4) \qquad \left[\frac{1}{2} g^{\beta i}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) \right] \frac{\partial y^b}{\partial x^\beta} = \frac{\partial^2 y^b}{\partial x^j \partial x^k}, \quad b, j, k = 1, \dots, n$$

NOTATION AND TERMINOLOGY :

CHRISTOFFEL SYMBOLS : $$\Gamma^{\beta}_{jk} = \frac{1}{2} g^{\beta i} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

$$\beta, j, k = 1, \dots, n$$

SO (4) CAN BE WRITTEN

$$\frac{\partial^2 y^b}{\partial x^j \partial x^k} = \Gamma^{\beta}_{jk} \frac{\partial y^b}{\partial x^\beta} \qquad b, j, k = 1, \dots, n \tag{5}$$

NEXT, FIX AN INDEX $b = 1, \dots, n$ AND DEFINE

$$z = (z_1, \dots, z_n) = \left(\frac{\partial y^b}{\partial x^1}, \dots, \frac{\partial y^b}{\partial x^n} \right)$$

SO THAT (5) CAN BE WRITTEN

$$\frac{\partial z_j}{\partial x^k} = \Gamma^{\beta}_{jk} z_\beta \quad , \quad j, k = 1, \dots, n$$

FOR EACH j, k AND ℓ WE MUST HAVE

$$\frac{\partial^2 z_j}{\partial x^\ell \partial x^k} = \frac{\partial^2 z_j}{\partial x^k \partial x^\ell}$$

SO

$$\frac{\partial}{\partial x^\ell} \left(\frac{\partial z_j}{\partial x^k} \right) = \frac{\partial}{\partial x^k} \left(\frac{\partial z_j}{\partial x^\ell} \right)$$

$$\frac{\partial}{\partial x^\ell} \left(\Gamma^{\beta}_{jk} z_\beta \right) = \frac{\partial}{\partial x^k} \left(\Gamma^{\beta}_{j\ell} z_\beta \right)$$

$$T^{\beta}_{jk}\,\frac{\partial z_\beta}{\partial x^l} + \frac{\partial T^{\beta}_{jk}}{\partial x^l}\,z_\beta = T^{\beta}_{jl}\,\frac{\partial z_\beta}{\partial x^k} + \frac{\partial T^{\beta}_{jl}}{\partial x^k}\,z_\beta$$

$$T^{\beta}_{jk}\,(T^{\gamma}_{\beta l}\,z_\gamma) + \underbrace{\frac{\partial T^{\beta}_{jk}}{\partial x^l}\,z_\beta}_{\frac{\partial T^{\gamma}_{jk}}{\partial x^l}\,z_\gamma} = T^{\beta}_{jl}\,(T^{\gamma}_{\beta k}\,z_\gamma) + \underbrace{\frac{\partial T^{\beta}_{jl}}{\partial x^k}\,z_\beta}_{\frac{\partial T^{\gamma}_{jl}}{\partial x^k}\,z_\gamma}$$

$$\left(\frac{\partial T^{\gamma}_{jk}}{\partial x^l} + T^{\beta}_{jk}\,T^{\gamma}_{\beta l} - \frac{\partial T^{\gamma}_{jl}}{\partial x^k} - T^{\beta}_{jl}\,T^{\gamma}_{\beta k}\right) z_\gamma = 0$$

<u>NOTATION</u> : FOR EACH $\gamma, j, k, l = 1, \ldots, n$ LET

$$\boxed{R^{\gamma}_{jkl} = \frac{\partial T^{\gamma}_{jk}}{\partial x^l} + T^{\gamma}_{\beta l}\,T^{\beta}_{jk} - \frac{\partial T^{\gamma}_{jl}}{\partial x^k} - T^{\gamma}_{\beta k}\,T^{\beta}_{jl}}$$

AND NOTE THAT THESE ARE ENTIRELY DETERMINED BY THE METRIC COMPONENTS (g_{ij}) IN THE CHART (U, φ).

$$R^{\gamma}_{ijk}\,z_\ell = 0 \qquad i, j, k = 1, \ldots, n$$

$$R^{\gamma}_{ijk}\,\frac{\partial y^b}{\partial x^\gamma} = 0 \qquad b, i, j, k = 1, \ldots, n$$

FOR ANY FIXED $i, j, k = 1, \ldots, n$ THIS CAN BE REGARDED AS A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS IN $(R^{1}_{ijk}, \ldots, R^{n}_{ijk})$ AND, SINCE THE JACOBIAN $\left(\frac{\partial y^b}{\partial x^\gamma}\right)$ IS NONSINGULAR, WE CONCLUDE THAT

$$R^{\gamma}_{ijk} = 0 \qquad \gamma, i, j, k = 1, \ldots, n$$

WE HAVE SHOWN THAT THE EXISTENCE OF A CHART (V, ψ) IN WHICH THE METRIC COMPONENTS ARE δ_{ab} EVERYWHERE ON V IMPLIES THAT $R^{\,r}_{ijk} = 0$ EVERYWHERE ON V.

INSTEAD OF REGARDING SUCH A LOCAL COORDINATE SYSTEM AS BEING GIVEN ONE CAN REGARD

$$(1) \qquad \sum_{a=1}^{n} \frac{\partial y^a}{\partial x^i} \frac{\partial y^a}{\partial x^j} = g_{ij}$$

AS A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS FOR SUCH A SET $y^1, \ldots, y^n$ OF COORDINATES.

IT CAN BE SHOWN THAT $R^{\,r}_{ijk} = 0$ IS NOT ONLY NECESSARY, BUT ALSO SUFFICIENT FOR THE EXISTENCE OF A SOLUTION TO THESE EQUATIONS

CONSEQUENCE : THE CONDITION THAT $R^{\,r}_{ijk} = 0$ ON AN OPEN SET IN M IS INDEPENDENT OF THE CHOICE OF COORDINATES.

SO, WHAT IS THE POINT OF THIS LONG AND ANNOYING CALCULATION ? WHAT WE HAVE DONE IS ARRIVE AT n^4 RIDICULOUSLY COMPLICATED FUNCTIONS $R^{\,r}_{jkl}$ COMPUTABLE FROM THE METRIC COMPONENTS g_{ij} IN SOME CHART (U, φ), THE VANISHING OF WHICH ON U IS EQUIVALENT TO AN AFFIRMATIVE ANSWER TO THE QUESTION :

" IS THERE ANOTHER COORDINATE SYSTEM $y^1, \dots, y^n$ ON U

IN WHICH THE METRIC IS 'FLAT' : $dy^1 \otimes dy^1 + \dots + dy^n \otimes dy^n$? "

THE FUNCTIONS R^{r}_{jkl} ARE KNOWN AS THE COMPONENTS OF THE " RIEMANN CURVATURE TENSOR " OF (M, g) IN THE CHART (U, φ).

THIS, HOWEVER, IS THE BEGINNING OF ANOTHER STORY AND OUR TIME APPEARS TO BE OVER.