BEFORE PROVING ANY GENERAL RESULTS WE WILL TRY TO MOTIVATE WHAT IS TO COME BY LOOKING AT THE EXAMPLE

$$SO(3) = \text{LIE GROUP OF ALL } 3\times 3 \text{ REAL MATRICES } g = (g_{ij}) \text{ SATISFYING } gg^T = g^Tg = id \text{ AND } \det(g) = 1.$$

$SO(3)$ IS A SUBMANIFOLD OF $\mathbb{R}^9$ SO ITS TANGENT SPACE

$$T_{id}(SO(3))$$

AT THE IDENTITY $id \in SO(3)$ CAN BE IDENTIFIED WITH A LINEAR SUBSPACE OF $\mathbb{R}^9$ (I.E., WITH SOME SET OF 3×3 MATRICES).

$$T_{id}(SO(3)) = \{\alpha'(0) : \alpha \text{ IS A SMOOTH CURVE IN } SO(3) \text{ WITH } \alpha(0) = id\}$$

HERE ARE SOME EXAMPLES OF SUCH CURVES: $\alpha_i : (-\infty,\infty) \to SO(3)$, $i = 1,2,3$:

$$\alpha_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \qquad \alpha_2(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}$$

$$\alpha_3(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

NOTE THAT, FOR EACH $i = 1, 2, 3$, AND, FOR EACH t, $\alpha_i(t) \in SO(3)$ AND $\alpha_i(0) = id$.

SINCE $SO(3)$ IS A SUBMANIFOLD OF $\mathbb{R}^9$, $\alpha_i'(t)$ CAN BE COMPUTED BY COORDINATEWISE (I.E., ENTRYWISE) DIFFERENTIATION WITH RESPECT TO t. THIS GIVES

$$\alpha_1'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \alpha_2'(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\alpha_3'(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

THESE ARE ALL IN $T_{id}(SO(3))$ AND, AS ELEMENTS OF $\mathbb{R}^9$, LINEARLY INDEPENDENT. SINCE $\dim T_{id}(SO(3)) = 3$, THESE FORM A BASIS FOR $T_{id}(SO(3))$. THUS, $T_{id}(SO(3))$ IS THE SET OF ALL

$$\sum_{i=1}^{3} n^i \alpha_i'(0) = \begin{pmatrix} 0 & -n^3 & n^2 \\ n^3 & 0 & -n^1 \\ -n^2 & n^1 & 0 \end{pmatrix}$$

FOR $n^1, n^2, n^3 \in \mathbb{R}$.

THESE ARE PRECISELY THE 3×3 REAL SKEW-SYMMETRIC MATRICES.

$T_{id}(SO(3))$ = VECTOR SPACE OF ALL 3×3 REAL MATRICES A THAT ARE SKEW-SYMMETRIC ($A^T = -A$)

NOW NOTICE THE FOLLOWING :

IF A AND B ARE SKEW-SYMMETRIC MATRICES, THEN SO IS THEIR COMMUTATOR

$$[A,B] = AB - BA$$

<u>PROOF</u> : $[A,B]^T = (AB-BA)^T = (AB)^T - (BA)^T =$
$B^TA^T - A^TB^T = (-B)(-A) - (-A)(-B) =$
$BA - AB = -(AB-BA) = -[A,B]$. $\square$

THUS, $T_{id}(SO(3))$ IS CLOSED UNDER FORMATION OF COMMUTATORS.

WE WILL SEE LATER THAT THIS IS TRUE FOR <u>EVERY</u> MATRIX LIE GROUP.

THUS, WE HAVE AN EXAMPLE OF THE FOLLOWING :

A REAL VECTOR SPACE V IS A <u>LIE</u> <u>ALGEBRA</u> IF THERE IS DEFINED ON IT AN OPERATION

$$[\ ,\ ] : V \times V \to V$$

(CALLED <u>BRACKET</u>) THAT SATISFIES

1. (BILINEARITY) $[av + bw, u] = a[v,u] + b[w,u]$
$$[v, aw + bu] = a[v,w] + b[v,u]$$

2. (SKEW-SYMMETRY) $[w, v] = -[v, w]$

3. (JACOBI IDENTITY)

$$[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$$

<u>NOTES</u>: THE MATRIX COMMUTATOR IS EASILY SEEN TO SATISFY ALL OF THESE. RECALL THAT THE COLLECTION OF ALL SMOOTH VECTOR FIELDS ON A MANIFOLD IS A LIE ALGEBRA UNDER THE LIE BRACKET.

WHEN EQUIPPED WITH THE LIE ALGEBRA STRUCTURE PROVIDED BY THE COMMUTATOR IT IS CUSTOMARY TO WRITE $T_{id}(SO(3))$ AS

$$so(3)$$

AND CALL IT THE <u>LIE</u> <u>ALGEBRA</u> <u>OF</u> <u>SO(3)</u>.

TO EXPLORE THE RELATIONSHIP BETWEEN $SO(3)$ AND $\mathfrak{so}(3)$ WE NEED SOME PROPERTIES OF MATRIX EXPONENTIATION :

FOR ANY $n \times n$ (REAL OR COMPLEX) MATRIX A ONE DEFINES

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = id + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \cdots$$

I WILL ASSUME THAT YOU HAVE SEEN THIS BEFORE AND THAT YOU KNOW THAT THE SERIES CONVERGES ENTRYWISE. IN FACT, THE SERIES CONVERGES ABSOLUTELY FOR ANY A AND UNIFORMLY ON ANY BOUNDED SET (IN $\mathbb{R}^{n^2}$ OR $\mathbb{C}^{n^2}$).

I WILL LET YOU LOOK UP (OR PROVE FOR YOURSELF) THE FOLLOWING BASIC PROPERTIES OF MATRIX EXPONENTIATION :

1. $\exp(A+B) = \exp(A)\exp(B)$ PROVIDED A AND B COMMUTE.
2. $\det(\exp(A)) = e^{Tr(A)}$
3. $\exp(\bar{A}^T) = \overline{\exp(A)}^T$

ALTHOUGH WE WON'T NEED IT, THERE IS AN IMPORTANT (AND CONSIDERABLY MORE DIFFICULT) GENERALIZATION OF #1 THAT YOU SHOULD AT LEAST SEE SO HERE IT IS :

<u>TROTTER</u> <u>PRODUCT</u> <u>FORMULA</u> : FOR ANY $n \times n$ MATRICES

$$\exp(A+B) = \lim_{n\to\infty} \left(\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\right)^n$$

LET'S DO A LITTLE CALCULATION :

LET $n = (n^1, n^2, n^3)$ BE A UNIT VECTOR IN $\mathbb{R}^3$:

$$(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$$

CONSIDER THE ELEMENT

$$N = \begin{pmatrix} 0 & -n^3 & n^2 \\ n^3 & 0 & -n^1 \\ -n^2 & n^1 & 0 \end{pmatrix}$$

OF $\mathfrak{so}(3)$.

<u>EXERCISE</u> : SHOW THAT $N^2 = \begin{pmatrix} -((n^2)^2+(n^3)^2) & n^1n^2 & n^1n^3 \\ n^1n^2 & -((n^1)^2+(n^3)^2) & n^2n^3 \\ n^1n^3 & n^2n^3 & -((n^1)^2+(n^2)^2) \end{pmatrix}$

AND

$$N^3 = -N, \quad N^4 = -N^2, \quad N^5 = N, \quad \ldots \quad \text{(REPEAT)}$$

AND USE THIS TO SHOW THAT, FOR ANY REAL NUMBER t,

$$\exp(tN) = \mathrm{id} + (\sin t)N + (1-\cos t)N^2$$

NOW NOTICE THAT

$$(\exp(tN))(\exp(tN))^T = (id + (\sin t)N + (1-\cos t)N^2)(id - (\sin t)N + (1-\cos t)N^2)$$

(N^2 IS SYMMETRIC)

$$= id - (\sin t)N + (1-\cos t)N^2 + (\sin t)N - (\sin^2 t)N^2 + (\sin t)(1-\cos t)(-N) + (1-\cos t)N^2 - (\sin t)(1-\cos t)(-N) + (1-\cos t)^2(-N^2)$$

$$= id + (1-\cos t - \sin^2 t + 1 - \cos t - (1-\cos t)^2)N^2$$

$$= id + (1-\cos t - \sin^2 t + 1 - \cos t - 1 + 2\cos t - \cos^2 t)N^2$$

$$= id$$

THUS,

$$\exp(tN) \in O(3)$$

MOREOVER,

$$\det(\exp(tN)) = e^{Tr(tN)} = e^0 = 1$$

SO

$$\exp(tN) \in SO(3)$$

<u>CONCLUSION</u>: $t \in \mathbb{R}$ AND $(n^1, n^2, n^3) \in \mathbb{R}^3$ WITH $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$.

LET

$$N = \begin{pmatrix} 0 & -n^3 & n^2 \\ n^3 & 0 & -n^1 \\ -n^2 & n^1 & 0 \end{pmatrix} \in so(3).$$

THEN

$$\exp(tN) \in SO(3)$$

A BIT MORE WORK WOULD SHOW THAT, IN FACT,

$\exp(tN)$ = ROTATION OF $\mathbb{R}^3$ THROUGH t RADIANS ABOUT THE AXIS IN $\mathbb{R}^3$ ALONG $\vec{n} = (n^1, n^2, n^3)$ IN THE SENSE DETERMINED BY THE RIGHT-HAND RULE FROM THE DIRECTION OF $\vec{n}$.

THE FOLLOWING THEOREM SHOULD THEREFORE COME AS NO SURPRISE

<u>THEOREM</u>: FOR ANY $t \in \mathbb{R}$ AND ANY $\vec{n} = (n^1, n^2, n^3) \in \mathbb{R}^3$ WITH $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$, $\exp(tN) \in SO(3)$. CONVERSELY, GIVEN ANY $R \in SO(3)$ THERE IS A UNIQUE REAL NUMBER t IN $[0, \pi]$ AND A UNIT VECTOR $\vec{n} = (n^1, n^2, n^3)$ IN $\mathbb{R}^3$ SUCH THAT

$$R = \exp(tN).$$

MOREOVER,

(a) IF t IS IN $(0, \pi)$, THEN $\vec{n}$ IS UNIQUE

(b) IF $t = 0$, THEN ANY $\vec{n}$ WILL DO

(c) IF $t = \pi$, THEN $\vec{n}$ IS UNIQUE UP TO SIGN

THE PROOF OF THE CONVERSE IS NOT DIFFICULT, BUT BASICALLY AMOUNTS TO SOLVING A LOT OF ALGEBRAIC EQUATIONS AND I DO NOT WANT TO SPEND TIME ON IT HERE (IF YOU WOULD LIKE TO SEE A PROOF I WILL SUPPLY A REFERENCE).

FROM THE THEOREM WE CONCLUDE THAT

$$\exp : \mathfrak{so}(3) \longrightarrow SO(3)$$

IS <u>SURJECTIVE</u>.

AS A SET OF MATRICES, AT LEAST, $SO(3)$ CAN BE COMPLETELY RETRIEVED FROM ITS LIE ALGEBRA $\mathfrak{so}(3)$.

MUCH MORE IS TRUE, HOWEVER. EVEN THE CHARTS ON $SO(3)$ CAN BE OBTAINED FROM $\mathfrak{so}(3)$!

TO SEE THIS, NOTE THE FOLLOWING :

> $SO(3)$ IS A SUBMANIFOLD OF $\mathbb{R}^9$ SO THE TANGENT SPACE $\mathfrak{so}(3)$ CAN BE IDENTIFIED WITH A (3-DIMENSIONAL) VECTOR SUBSPACE OF $\mathbb{R}^9$. THUS, $\mathfrak{so}(3)$ CAN ALSO BE THOUGHT OF AS A SUBMANIFOLD OF $\mathbb{R}^9$ AND
>
> $$\exp : \mathfrak{so}(3) \longrightarrow SO(3)$$
>
> IS A <u>SMOOTH</u> MAP BETWEEN THESE MANIFOLDS

THUS, WE CAN COMPUTE DERIVATIVES $\exp_{*p}$ FOR THIS MAP AT ANY $p \in so(3)$ (NOTE : $so(3) \cong \mathbb{R}^3$ SO $T_p(so(3)) \cong \mathbb{R}^3$ AS WELL).

TAKE $p = 0$ (THE 3×3 ZERO MATRIX IN $so(3)$). THEN $\exp(0) = id$ SO

$$\exp_{*0} : T_0(so(3)) \longrightarrow T_{id}(SO(3))$$

OR

$$\exp_{*0} : so(3) \longrightarrow so(3)$$

<u>CLAIM</u> : $\exp_{*0} = id_{so(3)}$ (SO, IN PARTICULAR, IT IS AN ISOMORPHISM)

<u>PROOF</u> : LET $A \in so(3)$. THEN $\alpha : (-\infty, \infty) \to so(3)$ GIVEN BY

$$\alpha(t) = 0 + tA = tA$$

IS A SMOOTH CURVE IN $so(3)$ WITH $\alpha(0) = 0$ AND $\alpha'(0) = A$.

THUS,

$$\begin{aligned}\exp_{*0}(A) = \exp_{*0}(\alpha'(0)) &= (\exp \circ \alpha)'(0) \\ &= \frac{d}{dt}(\exp(\alpha(t)))\Big|_{t=0} \\ &= \frac{d}{dt}(\exp(tA))\Big|_{t=0} \\ &= \frac{d}{dt}(id + tA + \tfrac{1}{2}t^2A^2 + \cdots)\Big|_{t=0} \\ &= A\end{aligned}$$

$\square$

THE INVERSE FUNCTION THEOREM THEREFORE IMPLIES THAT

$$\exp : \mathfrak{so}(3) \longrightarrow SO(3)$$

IS A LOCAL DIFFEOMORPHISM NEAR $0 \in \mathfrak{so}(3)$. SINCE $\exp(0) = id \in SO(3)$ WE CAN FIND AN OPEN NEIGHBORHOOD U OF id IN $SO(3)$ ON WHICH $\exp^{-1}$ EXISTS AND IS SMOOTH AND MAPS ONTO AN OPEN NEIGHBORHOOD OF 0 IN $\mathfrak{so}(3) \cong \mathbb{R}^3$. THUS,

$$(U, \exp^{-1})$$

IS A <u>CHART</u> AT id IN $SO(3)$.

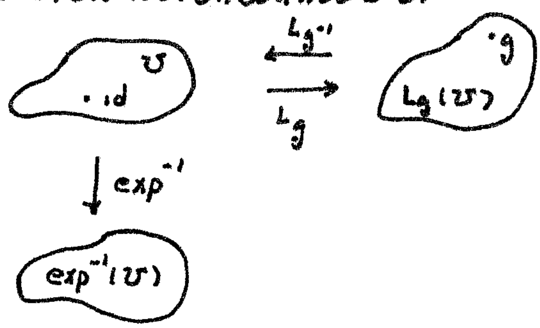
NOW NOTICE THAT THE CHART AT id IN $SO(3)$ GIVES RISE TO A CHART AT ANY g IN $SO(3)$ BECAUSE

$$L_g : SO(3) \longrightarrow SO(3)$$
$$L_g(h) = gh$$

IS A DIFFEOMORPHISM (SO $L_g(U)$ IS AN OPEN NEIGHBORHOOD OF g AND

$$\exp^{-1} \circ L_{g^{-1}}$$

A CHART MAP)



BASICALLY EVERYTHING ABOUT THE LIE GROUP $SO(3)$ CAN BE RECOVERED FROM THE LIE ALGEBRA $\mathfrak{so}(3)$.