MATH 680 LECTURE 24 SECTIONAL CURVATURE

BACK TO RIEMANNIAN n-MANIFOLDS. RECALL THAT FOR ANY SUCH (M, g) AND ANY $p \in M$

$$\exp_p : T_p(M) \to M$$

IS A DIFFEOMORPHISM OF AN OPEN NEIGHBORHOOD OF $0 \in T_p(M)$ ONTO AN OPEN NEIGHBORHOOD OF p IN M.

SUPPOSE σ IS A 2-DIMENSIONAL LINEAR SUBSPACE OF $T_p(M)$.

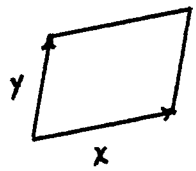
THE RESTRICTION OF $\exp_p$ TO σ CARRIES AN OPEN NEIGHBORHOOD OF 0 IN σ ONTO A 2-DIMENSIONAL SUBMANIFOLD Σ OF M CONTAINING p.

Σ INHERITS A RIEMANNIAN METRIC ($\iota^* g$, WHERE $\iota : \Sigma \hookrightarrow M$) FROM M AND SO IT HAS A GAUSSIAN CURVATURE WHICH WE WILL DENOTE $\mathcal{K}(\sigma)$.

THE FUNCTION WHICH ASSIGNS TO EACH 2-DIMENSIONAL LINEAR SUBSPACE σ OF $T_p(M)$ THIS GAUSSIAN CURVATURE $\mathcal{K}(\sigma)$ AT p BASICALLY MEASURES THE CURVATURE AT p OF EVERY 2-DIMENSIONAL SUCH "SLICE" OF M THROUGH p.

TO DO ALL OF THIS ANALYTICALLY WE WILL NEED A BIT OF LINEAR ALGEBRA.

RECALL : IN $\mathbb{R}^2$ WITH ITS USUAL INNER PRODUCT $\langle\ ,\ \rangle$,



$$\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2} \ = \text{ AREA OF THE PARALLELOGRAM SPANNED BY } x \text{ AND } y$$

THE SAME NOTION CAN BE INTRODUCED IN ANY VECTOR SPACE V WITH ANY INNER PRODUCT $\langle\ ,\ \rangle$.

EXAMPLE : IN A 2-DIMENSIONAL RIEMANNIAN MANIFOLD WITH x AND y TAKEN TO BE COORDINATE VELOCITY VECTORS ∂_1 AND ∂_2 SPANNING THE TANGENT SPACE AT SOME POINT THIS REDUCES TO

$$\sqrt{g_{11}g_{22} - g_{12}^2} = \sqrt{\det(g)}$$

WHICH WE HAVE HAD OCCASION TO SEE BEFORE.

ONE MORE OBSERVATION : LET (M,g) BE A RIEMANNIAN n-MANIFOLD WITH CURVATURE $R : \mathcal{T}(TM) \times \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow \mathcal{T}(TM)$

$$(X,Y,Z) \rightarrow R(X,Y)Z$$

AND, FOR CONVENIENCE, WRITE

$$[X,Y,Z,W] = g(R(X,Y)Z,W)$$

NOTE THAT THE VALUE OF THE VECTOR FIELD $R(X,Y)Z$ AT ANY $p \in M$ DEPENDS ONLY ON THE VALUES OF X, Y AND Z AT p BECAUSE LOCALLY WE CAN WRITE

$$X = X^i \partial_i$$
$$Y = Y^j \partial_j$$
$$Z = Z^k \partial_k$$

AND

$$\begin{aligned} R(X,Y)Z &= R(X^i \partial_i , Y^j \partial_j)(Z^k \partial_k) \\ &= X^i Y^j Z^k R(\partial_i, \partial_j)\partial_k \qquad (C^\infty(M)\text{-MODULE HOMOMORPHISM}) \\ &= X^i Y^j Z^k R^{\ell}_{\ ijk} \partial_\ell \end{aligned}$$

SO

$$(R(X,Y)Z)(p) = X^i(p) Y^j(p) Z^k(p) R^{\ell}_{\ ijk}(p) \, \partial_\ell|_p$$

THE SAME IS THEREFORE TRUE OF THE REAL-VALUED FUNCTION $[X,Y,Z,W]$, I.E., ITS VALUE AT p DEPENDS ONLY ON $X(p)$, $Y(p)$, $Z(p)$ AND $W(p)$.

<u>NOTE</u> : THIS CONTRASTS WITH COVARIANT DERIVATIVES $\nabla_V W$ (VALUE AT p DEPENDS ON $V(p)$ AND W <u>ON A NEIGHBORHOOD OF p</u>)

HERE'S WHAT THIS MEANS : GIVEN $x, y, z, w \in T_p(M)$ WE CAN DEFINE

$$[x, y, z, w] = [X, Y, Z, W](p)$$

WHERE X, Y, Z AND W ARE <u>ANY</u> VECTOR FIELDS DEFINED ON A NEIGHBORHOOD OF p WITH $X(p) = x$, $Y(p) = y$, $Z(p) = z$, $W(p) = w$.

<u>LEMMA</u>: LET (M, g) BE A RIEMANNIAN n-MANIFOLD WITH CURVATURE R AND LET $p \in M$. LET $\sigma \subseteq T_p(M)$ BE A 2-DIMENSIONAL LINEAR SUBSPACE OF $T_p(M)$ AND LET $x, y \in \sigma$ BE TWO LINEARLY INDEPENDENT VECTORS IN σ. THEN

$$K(x,y) = \frac{[x, y, x, y]}{\|x\|^2 \|y\|^2 - \langle x, y \rangle^2}$$

DOES NOT DEPEND ON THE CHOICE OF THE VECTORS x AND y (WHERE $\|x\|^2 = g_p(x,x)$, ETC.)

<u>PROOF</u>: ANY OTHER BASIS $\{x', y'\}$ FOR σ CAN BE GOTTEN FROM $\{x, y\}$ BY A NONSINGULAR LINEAR TRANSFORMATION IN σ AND ANY SUCH CAN BE OBTAINED AS A COMPOSITION OF MAPS OF THE FOLLOWING TYPE (FOR SOME NONZERO $\lambda \in \mathbb{R}$):

1. $\{x, y\} \to \{y, x\}$
2. $\{x, y\} \to \{\lambda x, y\}$
3. $\{x, y\} \to \{x + \lambda y, y\}$

IT'S EASY TO SEE THAT EACH OF THESE LEAVES $K(x,y)$ INVARIANT, E.G., FOR $\{x,y\} \to \{\lambda x, y\}$

$$\|\lambda x\|^2 \|y\|^2 - \langle \lambda x, y\rangle^2 = g_p(\lambda x, \lambda x)\, g_p(y,y) - (g_p(\lambda x, y))^2$$
$$= \lambda^2 \|x\|^2 \|y\|^2 - \lambda^2 \langle x,y\rangle^2$$
$$= \lambda^2 (\|x\|^2 \|y\|^2 - \langle x,y\rangle^2)$$

AND, EXTENDING x AND y TO VECTOR FIELDS X AND Y (SO THAT λX EXTENDS λx),

$$[\lambda x, y, \lambda x, y] = [\lambda X, Y, \lambda X, Y](p)$$
$$= g(R(\lambda X, Y)(\lambda X), Y)(p)$$
$$= g(\lambda^2 R(X,Y)X, Y)(p)$$
$$= \lambda^2 (g(R(X,Y)X, Y)(p))$$
$$= \lambda^2 [X, Y, X, Y](p)$$
$$= \lambda^2 [x, y, x, y]$$

SO $K(\lambda x, y) = K(x,y)$.

<u>EXERCISE</u> : VERIFY THE SAME THING FOR #1 AND #3. $\square$

FOR ANY SUCH σ WE WILL DENOTE THE VALUE OF $K(x,y)$ FOR ANY BASIS $\{x,y\}$ FOR σ,

$$K(\sigma)$$

AND CALL IT THE <u>SECTIONAL CURVATURE OF σ AT</u> p.

> <u>NOTE</u>: WHEN M IS 2-DIMENSIONAL THIS REDUCES TO THE GAUSSIAN CURVATURE OF M AT p.

WHEN $\dim M > 2$ WE THINK OF K AS A REAL-VALUED FUNCTION ON THE SET OF ALL 2-DIMENSIONAL PLANES IN $T_p(M)$ FOR ALL $p \in M$.

BEING JUST A WHOLE LOT OF GAUSSIAN CURVATURES, K IS A BIT EASIER TO RELATE TO THAN THE RIEMANNIAN CURVATURE R.

MOREOVER, KNOWING $K(\sigma)$ $\forall\, \sigma \subseteq T_p(M)$ $\forall\, p \in M$ COMPLETELY DETERMINES R, AS WE WILL NOW SHOW.

THIS IS ACTUALLY A PURELY ALGEBRAIC CONSEQUENCE OF THE LINEARITY AND SYMMETRIES OF R SO I WILL PHRASE THE NEXT RESULT AS A THEOREM IN LINEAR ALGEBRA.

LEMMA : LET V BE A VECTOR SPACE OF DIMENSION ≥ 2 WITH AN INNER PRODUCT $\langle \, , \rangle$. LET $R : V \times V \times V \to V$ BE A TRI-LINEAR MAPPING

$$(x, y, z) \longrightarrow R(x,y)z$$

AND ASSUME THAT

$$[x, y, z, w] := \langle R(x,y)z, w \rangle$$

SATISFIES

(a) $[x, y, z, w] = -[y, x, z, w]$

(b) $[x, y, z, w] = -[x, y, w, z]$

(c) $[x, y, z, w] = [z, w, x, y]$

(d) $[x, y, z, w] + [z, x, y, w] + [y, z, x, w] = 0$

FOR ANY 2-DIMENSIONAL SUBSPACE σ OF V DEFINE

$$K(\sigma) = \frac{[x, y, x, y]}{\|x\|^2 \|y\|^2 - \langle x, y \rangle^2}$$

WHERE $\{x, y\}$ IS ANY BASIS FOR σ. THEN K COMPLETELY DETERMINES R IN THE FOLLOWING SENSE : IF $R' : V \times V \times V \to V$ IS ANOTHER TRI-LINEAR MAP WITH THE SAME PROPERTIES AS R AND K' IS DEFINED FROM R' JUST AS K IS DEFINED FROM R, THEN

$$K'(\sigma) = K(\sigma) \;\forall \sigma \quad \Longrightarrow \quad R' = R .$$

PROOF: LET $[x,y,z,w]' = \langle R'(x,y)z, w \rangle$. BY NONDEGENERACY OF $\langle\ ,\ \rangle$ IT WILL BE ENOUGH TO SHOW THAT

$$[x,y,z,w] = [x,y,z,w]'$$

$\forall\ x,y,z,w \in V$.

BY HYPOTHESIS,

$$[x,y,x,y] = [x,y,x,y]'$$

FOR ALL LINEARLY INDEPENDENT $\{x,y\}$ IN V. SINCE (a) IMPLIES THAT

$$[x,x,z,w] = 0 \quad \forall\ z,w \in V$$

AND SIMILARLY FOR $[x,x,z,w]'$,

$$[x,y,x,y] = [x,y,x,y]'$$

$\forall\ x,y \in V$.

IN PARTICULAR, $\forall\ x,y,z \in V$

$$[x+z,y,x+z,y] = [x+z,y,x+z,y]'$$

$$[x,y,x+z,y] + [z,y,x+z,y] = [x,y,x+z,y]' + [z,y,x+z,y]'$$

$$[x,y,x,y] + [x,y,z,y] +$$

$$[z,y,x,y] + [z,y,z,y]$$

$$= [x,y,x,y]' + [x,y,z,y]' +$$

$$[z,y,x,y]' + [z,y,z,y]'$$

$$[x,y,z,y] + [z,y,x,y] = [x,y,z,y]' + [z,y,x,y]'$$

$$2[x,y,z,y] = 2[x,y,z,y]' \qquad \text{by (c)}$$

$$[x,y,z,y] = [x,y,z,y]' \qquad \forall\, x,y,z \in V$$

THUS,

$$[x, y+w, z, y+w] = [x, y+w, z, y+w]' \quad \forall\, x,y,z,w \in V$$

<u>EXERCISE</u> : SHOW THAT THIS IMPLIES

$$[x,y,z,w] - [x,y,z,w]' = [y,z,x,w] - [y,z,x,w]'$$

$\forall\; x,y,z,w \in V$ AND CONCLUDE THAT THE EXPRESSION

$$[x,y,z,w] - [x,y,z,w]'$$

IS INVARIANT UNDER CYCLIC PERMUTATION OF THE FIRST THREE ENTRIES (I.E., IS THE SAME FOR x,y,z , z,x,y AND y,z,x).

THE SUM OF

$$[x,y,z,w] - [x,y,z,w]'$$

$$[z,x,y,w] - [z,x,y,w]'$$

$$[y,z,x,w] - [y,z,x,w]'$$

IS 0 BY (d) SO

$$3\left([x,y,z,w] - [x,y,z,w]'\right) = 0$$

SO

$$[x,y,z,w] = [x,y,z,w]'$$

$\forall\, x,y,z,w \in V$ AS REQUIRED $\square$

THE MOST NATURAL PLACE TO START LOOKING AT SECTIONAL CURVATURE IS THOSE RIEMANNIAN MANIFOLDS FOR WHICH IT IS CONSTANT.

A RIEMANNIAN MANIFOLD (M, g) IS SAID TO HAVE <u>CONSTANT CURVATURE</u> IF THERE EXISTS A CONSTANT K_0 SUCH THAT $K(\sigma) = K_0$ FOR EVERY 2-DIMENSIONAL PLANE σ IN EVERY TANGENT SPACE $T_p(M)$, $p \in M$.

IN DIMENSION 2, WHERE $K = \mathcal{K}$, WE ALREADY KNOW A NUMBER OF EXAMPLES :

$$\mathbb{R}^2 \quad : \quad K_0 = 0$$

$$S^2 \quad : \quad K_0 = 1$$

$$H^2 \quad : \quad K_0 = -1$$