THERE ARE HIGHER DIMENSIONAL ANALOGUES:

$$\mathbb{R}^n : \quad K_0 = 0$$

$$S^n : \quad K_0 = 1$$

$$H^n : \quad K_0 = -1$$

THE FIRST OF THESE IS CLEAR ENOUGH SINCE THE RIEMANN CURVATURE OF $\mathbb{R}^n$ (STANDARD METRIC) IS IDENTICALLY ZERO, BUT THE LAST TWO ARE NONTRIVIAL. WE'LL GIVE THE PROOF FOR H^n, FOR WHICH WE WILL USE THE "UPPER HALF-SPACE" MODEL.

$$H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n > 0\}$$

$$g = g_{ij}\, dx^i \otimes dx^j$$

$$g_{ij} = \frac{1}{(x^n)^2} \delta_{ij}$$

SO

$$g = \frac{1}{(x^n)^2} (dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n)$$

<u>NOTE</u>: IT'S ABOUT TIME TO INTRODUCE THE FOLLOWING NOTION. TWO RIEMANNIAN METRICS g AND $\tilde{g}$ ON AN n-MANIFOLD M ARE SAID TO BE <u>CONFORMAL</u> IF THERE EXISTS A POSITIVE, SMOOTH REAL-VALUED FUNCTION ON M, WRITTEN e^μ FOR SOME $\mu \in C^\infty(M)$ FOR CONVENIENCE, SUCH THAT

$$\tilde{g} = e^\mu g$$

THUS, THE METRIC ON H^n IS CONFORMAL TO THE USUAL RIEMANNIAN METRIC ON THE UPPER HALF-SPACE $\mathbb{R}^n_+ = \{(x^1,\dots,x^n) \in \mathbb{R}^n : x^n > 0\}$.

THIS NOTION SEPARATES THE COLLECTION OF RIEMANNIAN METRICS ON M INTO EQUIVALENCE CLASSES (<u>CONFORMAL</u> <u>CLASSES</u>) AND A BASIC PROBLEM IN GEOMETRY IS

> GIVEN (M, g) FIND A METRIC $\tilde{g}$ ON M IN THE SAME CONFORMAL CLASS THAT DOES SUCH AND SUCH (E.G., HAS CONSTANT CURVATURE, ...).

TO DO OUR CALCULATION WE FIRST NEED A CONVENIENT WAY OF RECOGNIZING SPACES OF CONSTANT CURVATURE.

<u>LEMMA</u>: LET (M, g) BE A RIEMANNIAN MANIFOLD. IF (M, g) HAS CONSTANT SECTIONAL CURVATURE K_0, THEN FOR ANY VECTOR FIELDS $X, Y, Z, W \in \Gamma(TM)$,

$$[X,Y,Z,W] = K_0\big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big)$$

I.E.,

$$g(R(X,Y)Z,W) = K_0\big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big)$$

CONVERSELY, IF THIS EQUALITY IS SATISFIED $\forall X, Y, Z, W$, THEN $K \equiv K_0$.

<u>PROOF</u> : OUR ASSUMPTION THAT THE SECTIONAL CURVATURE IS K_0 CAN BE WRITTEN

$$[X,Y,X,Y] = K_0\,(g(X,X)\,g(Y,Y) - (g(X,Y))^2)$$

$\forall\, X,Y \in \mathcal{T}^1(\mathfrak{M})$.

<u>EXERCISE</u> : DEFINE $R'(X,Y)Z$ BY THE REQUIREMENT THAT $\forall\, W$

$$g(R'(X,Y)Z, W) = g(X,Z)\,g(Y,W) - g(Y,Z)\,g(X,W)$$

AND SET $[X,Y,Z,W]' = g(R'(X,Y)Z, W)$. SHOW THAT $[\quad]'$ IS LINEAR IN EACH SLOT AND SATISFIES THE SYMMETRIES (a), (b), (c), (d) OF THE PREVIOUS LEMMA.

THUS, THE SAME IS TRUE OF

$$K_0\,[X,Y,Z,W]'.$$

BUT, BY ASSUMPTION,

$$[X,Y,X,Y] = K_0\,[X,Y,X,Y]'$$

SO THE PREVIOUS LEMMA GIVES

$$[X,Y,Z,W] = K_0\,[X,Y,Z,W]'$$

$\forall\, X,Y,Z,W$. THE CONVERSE IS CLEAR. $\square$

IN LOCAL COORDINATES THIS CRITERION FOR CONSTANT CURVATURE K_0 BECOMES

$$g(R(\partial_i,\partial_j)\partial_k,\partial_l) = K_0(g_{ik}g_{jl} - g_{il}g_{jk})$$

$$R_{ijkl} = K_0(g_{ik}g_{jl} - g_{il}g_{jk})$$

IN PARTICULAR, IF WE HAPPEN TO HAVE A CHART FOR WHICH THE COORDINATE VELOCITY VECTORS ARE ORTHOGONAL, THEN R_{ijkl} WILL BE ZERO UNLESS EITHER $i=k$ AND $j=l$, OR $i=l$ AND $j=k$, I.E., THE ONLY POTENTIALLY NONZERO CURVATURE COMPONENTS ARE

$$R_{ijij} \quad \text{AND} \quad R_{ijji} = -R_{ijij}$$

AT ANY FIXED POINT $p \in M$ WE CAN ALWAYS FIND COORDINATES FOR WHICH THE BASIS $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1,\dots,n}$ FOR $T_p(M)$ IS ORTHONORMAL (HOW?) AND, FOR THESE,

$$R_{ijkl}(p) = K_0(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

I.E.,

$$R_{ijij}(p) = -R_{ijji}(p) = K_0$$

AND

$$R_{ijkl}(p) = 0$$

IN OTHER CASES.

NOW WE JUST PROCEED TO COMPUTE THE R_{ijkl} IN STANDARD COORDINATES ON H^n (NOTICE THAT THESE ARE ORTHOGONAL).

$$g_{ij} = \frac{1}{(x^n)^2}\,\delta_{ij}$$

SO THE ONLY NONZERO PARTIAL DERIVATIVES ARE WITH RESPECT TO x^n AND

$$\frac{\partial g_{ij}}{\partial x^n} = -\frac{2}{(x^n)^3}\,\delta_{ij}$$

CHRISTOFFEL SYMBOLS :

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{i\alpha}\left(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\alpha}\right)$$

$$= \tfrac{1}{2} g^{ii}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

$$= \tfrac{1}{2} (x^n)^2\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

SO, IF <u>ALL</u> <u>THREE</u> <u>INDICES</u> ARE <u>DISTINCT</u>, $\Gamma^i_{jk} = 0$.

$$\Gamma^i_{ik} = \tfrac{1}{2}(x^n)^2\left(\frac{\partial g_{ki}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^i}\right)$$

$$= \tfrac{1}{2}(x^n)^2\,\frac{\partial g_{ii}}{\partial x^k}$$

$$\Gamma^i_{ik} = \begin{cases} 0 & \text{IF } k \neq n \\ -\dfrac{1}{x^n} & \text{IF } k = n \end{cases} = \Gamma^i_{ki}$$

$$\Gamma^{i}_{jj} = \frac{1}{2}(x^n)^2\left(\frac{\partial g_{ji}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i}\right) \qquad (\text{FOR } j \neq i)$$

$$= \frac{1}{2}(x^n)^2\left(-\frac{\partial g_{jj}}{\partial x^i}\right)$$

$$= \begin{cases} 0 & \text{IF } i \neq n \\ \dfrac{1}{x^n} & \text{IF } i = n \end{cases}$$

NONZERO CHRISTOFFEL SYMBOLS :

$$\Gamma^{i}_{in} = \Gamma^{i}_{ni} = -\frac{1}{x^n}, \qquad i = 1, \dots, n$$

$$\Gamma^{n}_{jj} = \frac{1}{x^n}, \qquad j = 1, \dots, n-1$$

THE ONLY NONZERO PARTIAL DERIVATIVES OF THE CHRISTOFFEL SYMBOLS ARE

$$\frac{\partial \Gamma^{i}_{in}}{\partial x^n} = \frac{\partial \Gamma^{i}_{ni}}{\partial x^n} = \frac{1}{(x^n)^2}, \qquad i = 1, \dots, n$$

$$\frac{\partial \Gamma^{n}_{jj}}{\partial x^n} = -\frac{1}{(x^n)^2}, \qquad j = 1, \dots, n$$

CURVATURE COMPONENTS :

$$R_{ijk\ell} = R^{\beta}_{ijk}\, g_{\beta\ell} = R^{\ell}_{ijk}\, g_{\ell\ell} = \frac{1}{(x^n)^2} R^{\ell}_{ijk}$$

$$R_{ijkl} = \frac{1}{(x^n)^2}\left[\frac{\partial T^l_{ik}}{\partial x^j} - \frac{\partial T^l_{jk}}{\partial x^i} + T^l_{j\alpha} T^\alpha_{ik} - T^l_{i\alpha} T^\alpha_{jk} \right]$$

EXERCISE : USE THE INFORMATION ON THE CHRISTOFFEL SYMBOLS DERIVED ABOVE TO SHOW THAT IF i, j, k AND l ARE ALL DISTINCT, THEN $R_{ijkl} = 0$. NOTE THAT IN THIS CASE $g_{ik}g_{jl} - g_{il}g_{jk} = 0$ AS WELL.

NEXT WE'LL LOOK AT

$$R_{ijij} = \frac{1}{(x^n)^2}\left[\frac{\partial T^j_{ii}}{\partial x^j} - \frac{\partial T^j_{ji}}{\partial x^i} + T^j_{j\alpha} T^\alpha_{ii} - T^j_{i\alpha} T^\alpha_{ji} \right]$$

WITH $i \neq j$ (R_{iiii} AND $g_{ii}g_{ii} - g_{ii}g_{ii}$ OBVIOUSLY BOTH ZERO).

NOTICE THAT IF NEITHER i NOR j IS n, THEN

$$\begin{aligned} R_{ijij} &= \frac{1}{(x^n)^2}\left[0 - 0 + T^j_{jn} T^n_{ii} - T^j_{in} T^n_{ji} \right] \\ &= \frac{1}{(x^n)^2}\left[\left(-\frac{1}{x^n}\right)\left(\frac{1}{x^n}\right) - 0 \right] \\ &= -\frac{1}{(x^n)^4} \end{aligned}$$

AND

$$g_{ii}g_{jj} - g_{ij}g_{ji} = \left(\frac{1}{(x^n)^2}\right)\left(\frac{1}{(x^n)^2}\right) - (0)(0) = \frac{1}{(x^n)^4}$$

SO

$$R_{ijij} = (-1)(g_{ii}g_{jj} - g_{ij}g_{ji})$$

I'LL DO ONE MORE CASE AND THEN LEAVE THE REST FOR YOU (DO IT - IT'S GOOD FOR YOUR SOUL). I'LL COMPUTE R_{njnj} (WITH $j \neq n$).

$$R_{njnj} = \frac{1}{(x^n)^2}\left[\frac{\partial \Gamma^j_{nn}}{\partial x^j} - \frac{\partial \Gamma^j_{jn}}{\partial x^n} + \Gamma^j_{j\alpha}\Gamma^\alpha_{nn} - \Gamma^j_{n\alpha}\Gamma^\alpha_{jn}\right]$$

$$= \frac{1}{(x^n)^2}\left[0 - \frac{1}{(x^n)^2} + \Gamma^j_{jn}\Gamma^n_{nn} - \Gamma^j_{nj}\Gamma^j_{jn}\right]$$

$$= \frac{1}{(x^n)^2}\left[\quad - \frac{1}{(x^n)^2} + \left(-\frac{1}{x^n}\right)\left(-\frac{1}{x^n}\right) - \left(-\frac{1}{x^n}\right)\left(-\frac{1}{x^n}\right)\right]$$

$$= -\frac{1}{(x^n)^4}$$

AND

$$g_{nn}\,g_{jj} - g_{nj}\,g_{jn} = \left(\frac{1}{(x^n)^2}\right)\left(\frac{1}{(x^n)^2}\right) - (0)(0) = \frac{1}{(x^n)^4}$$

SO

$$R_{njnj} = (-1)(g_{nn}\,g_{jj} - g_{nj}\,g_{jn})$$

IN ALL OF THE CASES WE HAVE CHECKED (AND, IF THERE IS ANY JUSTICE IN THE WORLD, IN THE REST AS WELL),

$$R_{ijk\ell} = (-1)(g_{ik}\,g_{j\ell} - g_{i\ell}\,g_{jk})$$

SO H^n HAS CONSTANT SECTIONAL CURVATURE

$$K_0 = -1 .$$

<u>NOTE</u> : YOU SHOULD BE AWARE OF THE FACT THAT, IN THE INTEREST OF SAVING TIME, WE HAVE PROVED THIS IN THE LEAST ELEGANT WAY POSSIBLE.

BEFORE MOVING ON TO RICCI AND SCALAR CURVATURE I WILL JUST RECORD A FEW THINGS ONE CAN PROVE AND A FEW THINGS THAT NO ONE HAS YET BE ABLE TO PROVE.

1. AS WE DID FOR 2-MANIFOLDS, WE WILL SAY THAT A RIEMANNIAN n-MANIFOLD IS <u>SPHERICAL</u>, <u>HYPERBOLIC</u>, OR <u>FLAT</u>, RESPECTIVELY, IF IT IS LOCALLY ISOMETRIC TO S^n, H^n, OR $\mathbb{R}^n$, RESPECTIVELY.

   ALSO AS IN THE 2-DIMENSIONAL CASE ONE CAN SHOW THAT THIS IS THE CASE IF AND ONLY IF THE MANIFOLD HAS CONSTANT CURVATURE 1, -1, OR 0, RESPECTIVELY.

   COMPLETE RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE ARE SOMETIMES CALLED <u>SPACE FORMS</u>. THESE CAN BE CHARACTERIZED GROUP THEORETICALLY (THEY ARE ALL QUOTIENTS OF S^n, H^n, OR $\mathbb{R}^n$ BY SOME SUBGROUP OF THE GROUP OF ISOMETRIES OF S^n, H^n, OR $\mathbb{R}^n$ – WE HAVE NOT YET DISCUSSED WHAT THIS MEANS).

2. FORGETTING ABOUT CONSTANT SECTIONAL CURVATURE ONE CAN FOCUS ON RIEMANNIAN MANIFOLDS (M, g) FOR WHICH THE SECTIONAL CURVATURE IS ALWAYS OF ONE SIGN.

THIS IS A MUCH MORE DIFFICULT BUSINESS, ESPECIALLY IN THE POSITIVE CASE. THERE ARE, FOR EXAMPLE, TOPOLOGICAL RESTRICTIONS, E.G.,

> <u>THEOREM</u>: A COMPACT, CONNECTED, ORIENTABLE RIEMANNIAN MANIFOLD OF EVEN DIMENSION WITH STRICTLY POSITIVE SECTIONAL CURVATURE MUST BE SIMPLY CONNECTED.

E.G., FOR 2-MANIFOLDS, S^2 IS THE ONLY ONE.

IN FACT, THERE ARE PRECIOUS FEW MANIFOLDS OF POSITIVE CURVATURE KNOWN. IF YOU WANT TO BE FAMOUS YOU SHOULD ANSWER

<u>HOPF'S</u> <u>QUESTION</u>: DOES $S^2 \times S^2$ ADMIT A RIEMANNIAN METRIC OF POSITIVE CURVATURE.

3. RESTRICT ATTENTION TO 3-DIMENSIONAL MANIFOLDS. HERE THERE IS A FAMOUS THEOREM OF MOSTOW:

<u>THEOREM</u>: IF A COMPACT, CONNECTED, ORIENTABLE 3-MANIFOLD ADMITS A HYPERBOLIC STRUCTURE (RIEMANNIAN METRIC OF CONSTANT CURVATURE -1), THEN THAT STRUCTURE IS <u>UNIQUE</u>.

HERE'S THE SIGNIFICANCE OF THIS. RIEMANNIAN METRICS MEASURE LENGTHS OF CURVES AND THEREFORE VOLUMES OF 3-DIMENSIONAL REGIONS

UNIQUE HYPERBOLIC METRIC $\Rightarrow$ UNIQUE HYPERBOLIC VOLUME

SO THIS HYPERBOLIC VOLUME IS A TOPOLOGICAL INVARIANT OF SUCH MANIFOLDS

<u>SnapPea</u>

THE REAL STORY IN 3-MANIFOLDS IS THE <u>THURSTON</u> <u>GEOMETRIZATION CONJECTURE</u> AND THE RECENT WORK OF <u>PERELMAN</u>.