SERIES EXPANSIONS:

RECALL (FROM CALCULUS) THAT MANY OF THE ELEMENTARY FUNCTIONS HAVE "POWER SERIES EXPANSIONS", E.G.,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

FOR ALL $-\infty < x < \infty$.

THESE HAVE MANY USES, E.G., APPROXIMATIONS

$$\sin x \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

THEY ALSO CAN CLARIFY THE PROPERTIES OF CERTAIN FUNCTIONS THAT MIGHT OTHERWISE BE OBSCURE, E.G., $\frac{\sin x}{x}$ IS UNDEFINED AT $x = 0$, BUT

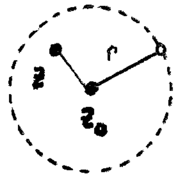
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots$$

SERIES EXPANSIONS PLAY AN EVEN MORE PROMINANT ROLE FOR COMPLEX FUNCTIONS. THE REASONS FOR THIS LIE IN INTEGRATION THEORY, WHICH WE TURN TO IN THE NEXT SECTION. HERE WE DESCRIBE ONLY THE BASIC THEORY, MUCH OF WHICH PARALLELS THE FAMILIAR MATERIAL FROM CALCULUS. BEGIN WITH A BIT OF NOTATION THAT WILL MAKE THINGS EASIER TO SAY.

FOR EACH COMPLEX NUMBER z_0 AND EACH POSITIVE REAL NUMBER r

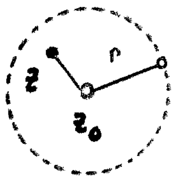
$U_r(z_0)$ = OPEN DISC OF RADIUS r ABOUT z_0

= ALL z IN $\mathbb{C}$ SATISFYING $|z - z_0| < r$



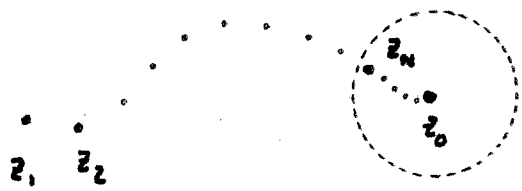
$U_r'(z_0)$ = DELETED OPEN DISC OF RADIUS r ABOUT z_0

= ALL z IN $\mathbb{C}$ SATISFYING $0 < |z - z_0| < r$



A SEQUENCE $(z_1, z_2, z_3, \ldots)$ OF COMPLEX NUMBERS IS SAID TO <u>CONVERGE</u> TO A COMPLEX NUMBER z_0 IF

FOR EACH $r > 0$ (HOWEVER SMALL) THERE IS AN N SUCH THAT z_n IS IN $U_r(z_0)$ FOR ALL $n \geq N$.



BEGINNING WITH ONE SEQUENCE $(z_1, z_2, z_3, \cdots)$ ONE CAN BUILD ANOTHER FROM IT BY SUCCESSIVELY ADDING TOGETHER MORE AND MORE OF ITS TERMS :

$$\begin{aligned} w_1 &= z_1 \\ w_2 &= z_1 + z_2 \\ &\vdots \\ w_n &= z_1 + z_2 + \cdots + z_n \\ &\vdots \end{aligned}$$

THIS IS CALLED AN <u>INFINITE</u> <u>SERIES</u> AND IS GENERALLY DENOTED

$$\sum_{n=1}^{\infty} z_n$$

(THEN $(w_n)_{n=1}^{\infty}$ IS CALLED ITS <u>SEQUENCE</u> <u>OF</u> <u>PARTIAL</u> <u>SUMS</u>).

E.G., LET z BE SOME FIXED COMPLEX NUMBER AND BEGIN WITH THE SEQUENCE $(1, z, z^2, \ldots, z^n, \ldots)$. THEN

$$w_n = 1 + z + z^2 + \cdots + z^{n-1} + z^n$$

NOTE THAT

$$z w_n = \quad z + z^2 + \cdots + z^n + z^{n+1}$$

SO

$$\begin{aligned} w_n - z w_n &= 1 - z^{n+1} \\ (1-z) w_n &= 1 - z^{n+1} \end{aligned}$$

AND, PROVIDED $z \neq 1$,

$$w_n = \frac{1 - z^{n+1}}{1 - z}$$

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

AS $n \to \infty$, $z^{n+1} \to 0$ IF AND ONLY IF $|z| < 1$ AND IT DIVERGES IF $|z| > 1$. SINCE $(w_n)_{n=1}^{\infty}$ CLEARLY DIVERGES IF $|z| = 1$ WE FIND THAT $\sum_{n=0}^{\infty} z^n$ CONVERGES TO $\frac{1}{1-z}$ IF $|z| < 1$ AND DIVERGES IF $|z| \geq 1$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad , \quad |z| < 1$$

(<u>GEOMETRIC</u> <u>SERIES</u>)

THIS IS A SPECIAL CASE OF

> A <u>POWER</u> <u>SERIES</u> <u>ABOUT</u> 0 IS AN INFINITE SERIES OF THE FORM
>
> $$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

WHICH IS ITSELF A SPECIAL CASE OF

> A <u>POWER</u> <u>SERIES</u> <u>ABOUT</u> z_0 IS AN INFINITE SERIES OF THE FORM
>
> $$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

NOTE THAT $\sum_{n=0}^{\infty} z^n$ ($a_n = 1$ FOR ALL n) CONVERGES FOR ALL z IN $U_1(0)$ AND ON THIS SET DEFINES AN ANALYTIC FUNCTION $\left(\frac{1}{1-z}\right)$.

THE GENERAL PATTERN IS THE SAME :

<u>THEOREM</u> : FOR ANY POWER SERIES $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ THERE IS AN $R \geq 0$ (POSSIBLY $R = \infty$) FOR WHICH

(a) THE SERIES CONVERGES (ABSOLUTELY) FOR EVERY z WITH

$$|z - z_0| < R$$

(b) THE SERIES DIVERGES FOR EVERY z WITH

$$|z - z_0| > R.$$

MOREOVER, ON $U_R(z_0)$ THE SUM OF THE SERIES IS AN ANALYTIC FUNCTION

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

WHOSE DERIVATIVE IS GIVEN ON $U_R(z_0)$ BY

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

(I.E., BY DIFFERENTIATING $f(z)$ "TERM-BY-TERM").

<u>NOTE</u> : THE PROOF OF THIS IS NOT SO BAD, BUT REQUIRES IDEAS FROM ANALYSIS THAT WE DO NOT ASSUME HERE. IF YOU WOULD LIKE TO SEE IT LOOK ON PAGES 39-40 OF <u>COMPLEX ANALYSIS</u>, LARS AHLFORS, McGRAW-HILL, 1966.

THE R IN THE THEOREM IS CALLED THE <u>RADIUS OF CONVERGENCE</u> OF THE POWER SERIES AND IT IS DETERMINED, AS IN CALCULUS, BY APPLYING THE "RATIO TEST".

SOME CONSEQUENCES :

A FUNCTION $f(z)$ DEFINED BY A POWER SERIES IS ANALYTIC WITHIN ITS RADIUS OF CONVERGENCE AND SO IS ITS DERIVATIVE $f'(z)$. BUT $f'(z)$ IS ALSO GIVEN BY A POWER SERIES SO IT IS ALSO ANALYTIC AND ITS DERIVATIVE $f''(z)$ CAN BE COMPUTED BY TERM-BY-TERM DIFFERENTIATION OF $f'(z)$. CONTINUING GIVES

$$f(z),\ f'(z),\ f''(z),\ \ldots,\ f^{(n)}(z),\ \ldots$$

ARE ALL ANALYTIC ON $U_R(z_0)$.

THE RATIO TEST APPLIED TO $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ EASILY SHOWS THAT $R = \infty$ SO THE SUM IS AN ENTIRE FUNCTION. BUT WHEN $z = x$ IS REAL THIS ENTIRE FUNCTION AGREES WITH $\sin x$. SINCE THIS UNIQUELY DEFINES $\sin z$ WE MUST HAVE

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad |z| < \infty$$

PRECISELY THE SAME ARGUMENT GIVES THE FOLLOWING COMPLEX VERSIONS OF ALL OF THE USUAL SERIES EXPANSIONS.

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad |z| < \infty$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \quad |z| < \infty$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \quad |z| < \infty$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \quad |z| < \infty$$

NOTE: THE MUCH MORE INTERESTING (AND MORE DIFFICULT) QUESTION OF WHETHER **EVERY** ANALYTIC FUNCTION HAS A POWER SERIES EXPANSION WILL HAVE TO WAIT UNTIL WE HAVE LEARNED SOMETHING ABOUT INTEGRATION.

EXERCISES:

50. USE SERIES EXPANSIONS TO SHOW THAT $\frac{d}{dz}(\sin z) = \cos z$.

51. USE THE GEOMETRIC SERIES TO SHOW FIRST THAT

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \quad |z| < 1,$$

AND THEN THAT

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}$$

AND

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

FOR $0 < r < 1$ AND ANY θ.

52. SHOW THAT

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

IS AN ENTIRE FUNCTION.

TWO POWER SERIES ABOUT z_0 THAT CONVERGE ON $|z - z_0| < R$ CAN BE ADDED (BY ADDING CORRESPONDING COEFFICIENTS) AND MULTIPLIED (BY THE DISTRIBUTIVE LAW) TO YIELD NEW SERIES THAT ALSO CONVERGE ON $|z - z_0| < R$. THEY CAN BE DIVIDED (BY LONG DIVISION) PROVIDED THE DENOMINATOR IS NONZERO ON $|z - z_0| < R$.

E.G.,

$$\begin{aligned} z^2 \sinh z &= z^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+3} \\ &= z^3 + \frac{1}{3!} z^5 + \frac{1}{5!} z^7 + \cdots \end{aligned}$$

ON ALL OF $\mathbb{C}$.

STILL MORE CONVERGENT SERIES EXPANSIONS CAN BE PRODUCED BY COMPOSITION,

E.G., ON $|z| < 1$,

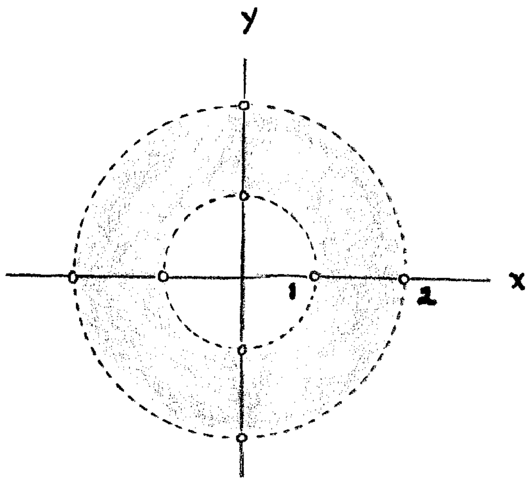
$$\begin{aligned} \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} &= \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} \\ &= 1 - z^2 + z^4 - z^6 + \cdots \end{aligned}$$

A CONVERGENT POWER SERIES REPRESENTS AN ANALYTIC FUNCTION ON A DISC.

MANY OF THE MOST IMPORTANT APPLICATIONS OF COMPLEX ANALYSIS DEPEND ON THE FACT ANALOGOUS EXPANSIONS (BUT WITH NEGATIVE AS WELL AS POSITIVE POWERS OF $z-z_0$) ARE POSSIBLE FOR CERTAIN FUNCTIONS THAT FAIL TO BE ANALYTIC. WE FIRST ILLUSTRATE WITH AN EXAMPLE.

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

FAILS TO BE ANALYTIC ONLY AT $z=1$ AND $z=2$. ON THE ANNULUS



$$1 < |z| < 2$$

$f(z)$ IS ANALYTIC AND HERE WE CAN WRITE (BECAUSE $|\frac{z}{2}| < 1$ AND $|\frac{1}{z}| < 1$)

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} \qquad \text{(PARTIAL FRACTIONS)}$$

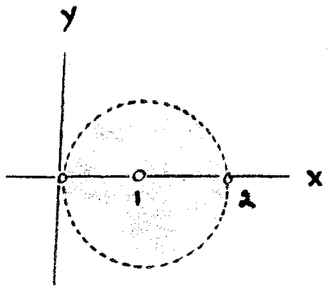
$$= \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})} = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\,\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n - \sum_{n=0}^{\infty} z^{-(n+1)}$$

$$= \sum_{n=1}^{\infty} -z^{-n} + \sum_{n=0}^{\infty}\frac{-1}{2^{n+1}}z^n$$

SIMILARLY, ON



$$0 < |z-1| < 1$$

(I.E., $U_1'(1)$, ALSO CALLED AN ANNULUS) WE CAN WRITE

$$f(z) = -\frac{1}{z-1} - \frac{1}{2-z} = -\frac{1}{z-1} - \frac{1}{1-(z-1)}$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$$

$$= -(z-1)^{-1} + \sum_{n=0}^{\infty} -(z-1)^n$$

<u>EXERCISE</u> 53 : FIND SIMILAR EXPANSIONS ON

(a) $0 < |z-2| < 1$ ANS. $(z-2)^{-1} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n$

(b) $|z| > 2$ ANS. $\sum_{n=0}^{\infty}(2^n-1)z^{-(n+1)}$

SERIES SUCH AS THESE ARE OF THE FORM

$$\cdots \underbrace{a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}$$

POWER SERIES IN $\frac{1}{z-z_0}$ CONVERGES ON SOME $|z-z_0| > r$

POWER SERIES IN $z-z_0$ CONVERGES ON SOME $|z-z_0| < R$

ARE CALLED <u>LAURENT SERIES</u> AND CONVERGE ON AN ANNULUS

$$r < |z-z_0| < R$$

AND ARE GENERALLY WRITTEN

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

(JUST A SHORTHAND NOTATION FOR THE SUM OF THE TWO SERIES ABOVE)

THE SUM OF A LAURENT SERIES REPRESENTS AN ANALYTIC FUNCTION ON ITS ANNULUS OF CONVERGENCE.

AS WE HAVE JUST SEEN IT IS OFTEN POSSIBLE TO SHOW THAT A FUNCTION WHICH FAILS TO BE ANALYTIC AT SOME z_0 CAN BE WRITTEN AS THE SUM OF VARIOUS LAURENT SERIES ON DIFFERENT ANNULI EXCLUDING z_0.

WE WILL LOOK AT A FEW MORE EXAMPLES.

<u>NOTE</u> : AS WITH FUNCTIONS THAT ARE ANALYTIC AT z_0, THE LARGER ISSUE OF PRECISELY WHICH FUNCTIONS HAVE SUCH EXPANSIONS WILL HAVE TO WAIT FOR SOME INTEGRATION THEORY.

1. $$f(z) = z^2 e^{\frac{1}{z}}$$

FAILS TO BE ANALYTIC AT $z_0 = 0$. ON $|z| > 0$,

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2-n}$$

$$= \cdots + \frac{1}{5!} z^{-3} + \frac{1}{3!} z^{-1} + \frac{1}{2} + z + z^2$$

2. $$f(z) = \frac{e^{-z}}{(z-1)^2}$$

FAILS TO BE ANALYTIC AT $z_0 = 1$. ON $0 < |z-1| < \infty$,

$$f(z) = \frac{e^{-1} e^{-(z-1)}}{(z-1)^2} = \frac{1}{e} \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} \frac{1}{n!} (-(z-1))^n$$

$$= \frac{1}{e} \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{e\, n!} (z-1)^{n-2}$$

$$= \frac{1}{e} (z-1)^{-2} - \frac{1}{e} (z-1)^{-1} + \sum_{n=2}^{\infty} \frac{(-1)^n}{e\, n!} (z-1)^{n-2}$$

3. $$f(z) = \frac{\sin z}{z}$$

FAILS TO BE ANALYTIC AT $z_0 = 0$. ON $|z| > 0$,

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

$$= 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots$$

EXAMPLE 1 : INFINITELY MANY NEGATIVE POWERS

2 : FINITELY MANY NEGATIVE POWERS

3 : NO NEGATIVE POWERS

SOME TERMINOLOGY :

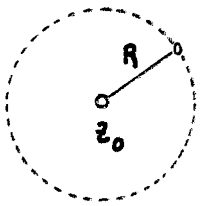
IF $f(z)$ FAILS TO BE ANALYTIC AT z_0, BUT IS ANALYTIC AT <u>SOME</u> POINT IN <u>EVERY</u> $U_r'(z_0)$, $r > 0$, THEN z_0 IS CALLED A <u>SINGULARITY</u> (OR <u>SINGULAR POINT</u>) OF $f(z)$.

A SINGULARITY z_0 OF $f(z)$ IS AN <u>ISOLATED SINGULARITY</u> IF IT IS ANALYTIC AT <u>EVERY</u> POINT IN <u>SOME</u> $U_R'(z_0)$

A NON-EXAMPLE : $z_0 = 0$ FOR $f(z) = \log z$

NOW ASSUME THAT, LIKE THE EXAMPLES ABOVE, $f(z)$ CAN BE WRITTEN AS THE SUM OF A LAURENT SERIES ON $U_R'(z_0)$

$0 < |z - z_0| < R$



<u>NOTE</u> : WE WILL SEE LATER THAT, IN FACT, IT <u>MUST</u> HAVE SUCH AN EXPANSION.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \underbrace{\cdots a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1}}_{\text{PRINCIPAL PART OF } f(z) \text{ AT } z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

z_0 IS AN <u>ESSENTIAL</u> <u>SINGULARITY</u> OF $f(z)$ IF THERE ARE INFINITELY MANY a_{-n}.

IF THERE IS A LARGEST m FOR WHICH $a_{-m} \neq 0$, z_0 IS A <u>POLE</u> <u>OF</u> <u>ORDER</u> m (POLE OF ORDER 1 IS A <u>SIMPLE</u> <u>POLE</u>).

z_0 IS A <u>REMOVABLE</u> <u>SINGULARITY</u> OF $f(z)$ IF $a_{-1} = a_{-2} = \cdots = 0$.

a_{-1} = <u>RESIDUE</u> <u>OF</u> $f(z)$ <u>AT</u> z_0 = $\operatorname*{Res}_{z=z_0} f(z)$

(THE SIGNIFICANCE OF THE RESIDUE WILL EMERGE IN INTEGRATION THEORY.)

<u>NOTE</u>: THE BEHAVIOR OF $f(z)$ NEAR AN ESSENTIAL SINGULARITY IS WILDLY COMPLICATED. HERE'S A (HARD) THEOREM WE WILL NOT PROVE:

<u>PICARD'S</u> <u>THEOREM</u>: IF $f(z)$ HAS AN ESSENTIAL SINGULARITY AT z_0, THEN IN EVERY $U_r'(z_0)$, HOWEVER SMALL, $f(z)$ TAKES EVERY COMPLEX VALUE, WITH ONE POSSIBLE EXCEPTION, INFINITELY OFTEN.

EXERCISES : FOR EACH OF THE FOLLOWING FUNCTIONS $f(z)$ FIND THE LAURENT SERIES EXPANSION ON $0 < |z - z_0| < R$ FOR THE GIVEN ISOLATED SINGULARITY z_0 (SPECIFY R). THEN CLASSIFY z_0 AS AN ESSENTIAL SINGULARITY, A POLE (SPECIFY THE ORDER), OR A REMOVABLE SINGULARITY. FINALLY, FIND $\operatorname*{Res}_{z=z_0} f(z)$.

54. $f(z) = \dfrac{1}{z^2 - 3z + 2}, \quad z_0 = 2.$

ANS. POLE OF ORDER 1

$\operatorname*{Res}_{z=2} f(z) = 1$

55. $f(z) = e^{\frac{1}{z}}, \quad z_0 = 0.$

ANS. ESSENTIAL SINGULARITY

$\operatorname*{Res}_{z=0} f(z) = 1$

56. $f(z) = \dfrac{1 - \cosh z}{z^3}, \quad z_0 = 0$

ANS. POLE OF ORDER 1

$\operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}$

57. $f(z) = \dfrac{e}{(z-1)^2}$

ANS. POLE OF ORDER 2

$\operatorname*{Res}_{z=1} f(z) = 2e^{2}$

58. SHOW THAT IF $f(z)$ HAS A REMOVABLE SINGULARITY AT z_0, THEN $\operatorname*{Res}_{z=z_0} f(z) = 0$.

SOLUTIONS TO THE EXERCISES :

50. $$\frac{d}{dz}(\sin z) = \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+1)z^{2n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$$

51. FOR $|z|<1$,
$$\frac{z}{1-z} = z\sum_{n=0}^{\infty}z^n = \sum_{n=0}^{\infty}z^{n+1} = \sum_{n=1}^{\infty}z^n$$

FOR $0<|z|<1$ SET $z=re^{i\theta}$. THEN
$$\sum_{n=1}^{\infty}z^n = \sum_{n=1}^{\infty}r^n e^{in\theta} = \sum_{n=1}^{\infty}r^n\cos n\theta + i\,r^n\sin n\theta$$
$$= \sum_{n=1}^{\infty}r^n\cos n\theta + i\sum_{n=1}^{\infty}r^n\sin n\theta$$

AND
$$\frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}} = \frac{r\cos\theta + i\,r\sin\theta}{(1-r\cos\theta)-i\,r\sin\theta}\;\frac{(1-r\cos\theta)+i\,r\sin\theta}{(1-r\cos\theta)+i\,r\sin\theta} =$$

$$\frac{[r\cos\theta(1-r\cos\theta)-r^2\sin^2\theta] + i[r\sin\theta(1-r\cos\theta)+r^2\sin\theta\cos\theta]}{(1-r\cos\theta)^2+r^2\sin^2\theta} =$$

$$\frac{[r\cos\theta - r^2] + i\,r\sin\theta}{1-2r\cos\theta+r^2} = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} + i\,\frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

SO EQUATING REAL AND IMAGINARY PARTS GIVES THE RESULTS.

52. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$ CONVERGES FOR ALL z AND SO REPRESENTS AN ENTIRE FUNCTION, ITS VALUE AT $z=0$ IS

$\frac{(-1)^0}{(2\cdot 0+1)!} z^{2\cdot 0} = 1$ AND, FOR $z \neq 0$, IT EQUALS

$$\frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{\sin z}{z} .$$

53. $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

(a) FOR $0 < |z-2| < 1$,

$$f(z) = \frac{1}{z-2} - \frac{1}{1+(z-2)} = \frac{1}{z-2} - \frac{1}{1-(-(z-2))}$$

$$= \frac{1}{z-2} - \sum_{n=0}^{\infty} (-(z-2))^n = \frac{1}{z-2} + \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n$$

$$= (z-2)^{-1} + \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n$$

(b) FOR $|z| > 2$, $\left|\frac{2}{z}\right| < 1$ (AND SO $\left|\frac{1}{z}\right| < 1$),

$$f(z) = \frac{1}{z} \frac{1}{1-\frac{2}{z}} - \frac{1}{z} \frac{1}{1-\frac{1}{z}}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} (2^n - 1) z^{-(n+1)}$$

54. FROM EXERCISE 53 (a),

$$f(z) = (z-2)^{-1} + \sum_{n=0}^{\infty} (-1)^{n+1}(z-2)^n$$

ON $0 < |z-2| < 1$ SO $z_0 = 2$ IS A POLE OF ORDER 1 WITH RESIDUE 1.

55. $$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

ON $0 < |z| < \infty$ SO $z_0 = 0$ IS AN ESSENTIAL SINGULARITY WITH RESIDUE $\frac{1}{1!} = 1$.

56. $$\frac{1-\cosh z}{z^3} = \frac{1 - \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}}{z^3} = \frac{1}{z^3}\sum_{n=1}^{\infty} -\frac{1}{(2n)!} z^{2n}$$

$$= \sum_{n=1}^{\infty} -\frac{1}{(2n)!} z^{2n-3}$$

$$= -\frac{1}{2!} z^{-1} + \sum_{n=2}^{\infty} -\frac{1}{(2n)!} z^{2n-3} \quad \text{ON } 0 < |z| < \infty$$

SO $z_0 = 0$ IS A POLE OF ORDER 1 WITH RESIDUE $-\frac{1}{2}$.

57. $$\frac{e^{2z}}{(z-1)^2} = \frac{e^{2(z-1)+2}}{(z-1)^2} = e^2\left(\frac{1}{(z-1)^2}\right)e^{2(z-1)}$$

$$= e^2\left(\frac{1}{(z-1)^2}\right)\sum_{n=0}^{\infty} \frac{1}{n!}(2(z-1))^n \quad \text{ON } 0 < |z-1| < \infty$$

$$= e^2 (z-1)^{-2} \sum_{n=0}^{\infty} \frac{2^n}{n!}(z-1)^n$$

$$= e^2 (z-1)^{-2} + 2e^2 (z-1)^{-1} + \sum_{n=2}^{\infty} \frac{2^n}{n!}(z-1)^n$$

SO $z_0 = 1$ IS A POLE OF ORDER 2 WITH RESIDUE $2e^2$.

58. FOR A REMOVABLE SINGULARITY, EVERY a_{-n} IS ZERO SO $\text{Res} = a_{-1} = 0$.