SERIES SOLUTIONS TO DIFFERENTIAL EQUATIONS

THE DIFFERENTIAL EQUATIONS WE HAVE SOLVED SO FAR ALL REQUIRED SOME SORT OF ALGEBRAIC " TRICK " FOR REWRITING THE EQUATION SO THAT WE CAN JUST INTEGRATE BOTH SIDES TO SOLVE (E.G., FOR LINEAR EQUATIONS SUCH AS $y' + 3y = e^{-2x}$, MULTIPLYING ON BOTH SIDES BY THE INTEGRATING FACTOR $\mu(x) = e^{3x}$ TURNS IT INTO $(e^{3x} y)' = e^{x}$, WHICH WE CAN INTEGRATE).

USUALLY SUCH TRICKS SIMPLY DON'T EXIST.

ALTERNATIVE : LOOK FOR A SOLUTION y GIVEN AS A POWER SERIES

$$y = \sum_{k=0}^{\infty} c_k x^k$$

BY JUST SUBSTITUTING THIS INTO THE EQUATION AND TRYING TO DETERMINE THE COEFFICIENTS $c_0, c_1, c_2, \ldots$

WE WILL WARM UP WITH A FEW SIMPLE EXAMPLES THAT WE ALREADY KNOW HOW TO SOLVE AND THEN DO ONE THAT WE COULD NOT SOLVE ANY OTHER WAY.

EXAMPLES :

1. $y' - y = 0$ (PRETEND THAT YOU DON'T KNOW HOW TO SOLVE THIS.)

LET $y = \sum_{k=0}^{\infty} c_k x^k$. THEN

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

RE-INDEX : $n = k-1$ (SO $k = n+1$)
$k = 1 \Rightarrow n = 0$

$$y' = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

NOW SUBSTITUTE BOTH OF THESE (y AND y') INTO THE EQUATION $y' - y = 0$ (USING n FOR THE NAME OF THE SUMMATION INDEX IN BOTH):

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1) c_{n+1} - c_n] x^n = 0$$

$\Rightarrow$

$$(n+1) c_{n+1} - c_n = 0 \quad \text{FOR ALL } n \geq 0$$

RECURRENCE RELATION: $c_{n+1} = \frac{1}{n+1} c_n$ FOR ALL $n \geq 0$

WRITE THESE OUT FOR THE FIRST FEW VALUES OF n:

$n = 0$: $\quad c_1 = \frac{1}{1} c_0$

$n = 1$: $\quad c_2 = \frac{1}{2} c_1 = \frac{1}{2} \cdot \frac{1}{1} c_0$

$n = 2$: $\quad c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} c_0$

$n = 3$: $\quad c_4 = \frac{1}{4} c_3 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} c_0$

$$\vdots$$

$$c_n = \frac{1}{n!} c_0 \qquad \text{FOR } n \geq 0$$

THUS,

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} c_0 x^n$$
$$= c_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

THERE IS NOTHING TO DETERMINE c_0 SO IT IS ARBITRARY. FOR ANY CHOICE OF c_0 THIS IS A SOLUTION TO THE EQUATION $y' - y = 0$.

NOTE : SINCE $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ THE ANSWER IS THE SAME AS WE WOULD HAVE GOTTEN EARLIER :

$$y = c_0 e^x \qquad (c_0 = \text{ARBITRARY CONSTANT})$$

2. $y'' + y = 0$ (KEEP PRETENDING)

LET $y = \sum_{k=0}^{\infty} c_k x^k$. THEN

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$

RE-INDEX: $n = k-2$ (SO $k = n+2$ AND $k-1 = n+1$)

$k = 2 \Rightarrow n = 0$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

NOW SUBSTITUTE BOTH y AND y'' INTO THE EQUATION $y'' + y = 0$ (USING n FOR THE NAME OF THE SUMMATION INDEX IN BOTH).

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + c_n] x^n = 0$$

$$\Rightarrow \quad (n+2)(n+1) c_{n+2} + c_n = 0 \quad \text{FOR ALL } n \geq 0$$

RECURRENCE RELATION:

$$c_{n+2} = -\frac{1}{(n+2)(n+1)} c_n \quad \text{FOR ALL } n \geq 0$$

WRITE THESE OUT FOR THE FIRST FEW VALUES OF n:

$n=0$: $c_2 = -\frac{1}{2\cdot 1} c_0$

$n=1$: $c_3 = -\frac{1}{3\cdot 2} c_1$

$n=2$: $c_4 = -\frac{1}{4\cdot 3} c_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1} c_0$

$n=3$: $c_5 = -\frac{1}{5\cdot 4} c_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2} c_1$

$\vdots$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x - \frac{1}{2!} c_0 x^2 - \frac{1}{3!} c_1 x^3 + \frac{1}{4!} c_0 x^4 + \frac{1}{5!} c_1 x^5 - \cdots$$

$$= c_0 \left[1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots \right] + c_1 \left[x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \right]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

THERE IS NOTHING TO DETERMINE c_0 AND c_1 SO THEY ARE ARBITRARY.

FOR ANY CHOICE OF c_0 AND c_1 THIS IS A SOLUTION TO $y'' + y = 0$

<u>NOTE</u>: SINCE $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$ AND $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin x$, THE ANSWER IS THE SAME AS WE WOULD HAVE GOTTEN EARLIER:

$$y = c_0 \cos x + c_1 \sin x$$

(c_0, c_1 ARBITRARY)

3. $y'' - xy' - y = 0$ (NO NEED TO PRETEND. YOU DON'T KNOW HOW TO SOLVE THIS ONE.)

LET $y = \sum_{k=0}^{\infty} c_k x^k$. THEN

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} \quad \text{AND} \quad xy' = x \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=1}^{\infty} k c_k x^k$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

NOW SUBSTITUTE ALL OF THESE INTO THE EQUATION (USING n AS THE SUMMATION INDEX IN ALL OF THEM).

$$y'' \quad - \quad xy' \quad - \quad y \quad = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$(0+2)(0+1) c_{0+2} x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n$$
$$- c_0 x^0 - \sum_{n=1}^{\infty} c_n x^n = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} - n c_n - c_n] x^n = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} - (n+1) c_n] x^n = 0$$

$$\Rightarrow \qquad 2c_2 - c_0 = 0$$

$$(n+2)(n+1) c_{n+2} - (n+1) c_n = 0 \qquad \text{FOR ALL } n \geq 0$$

$$\Rightarrow \qquad c_2 = \frac{1}{2} c_0$$

RECURRENCE RELATION: $c_{n+2} = \dfrac{1}{n+2} c_n$ FOR ALL $n \geq 0$

$$c_2 = \frac{1}{2} c_0$$

$n = 1$: $c_3 = \frac{1}{3} c_1$

$n = 2$: $c_4 = \frac{1}{4} c_2 = \frac{1}{2\cdot 4} c_0$

$n = 3$: $c_5 = \frac{1}{5} c_3 = \frac{1}{3\cdot 5} c_1$

$n = 4$: $c_6 = \frac{1}{6} c_4 = \frac{1}{2\cdot 4\cdot 6} c_0$

$n = 5$: $c_7 = \frac{1}{7} c_5 = \frac{1}{3\cdot 5\cdot 7} c_1$

$n = 6$: $c_8 = \frac{1}{8} c_6 = \frac{1}{2\cdot 4\cdot 6\cdot 8} c_0$

$\vdots$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + \cdots$$

$$= c_0 + c_1 x + c_0 x^2 + \frac{1}{3} c_1 x^3 + \frac{1}{2\cdot 4} c_0 x^4 + \frac{1}{3\cdot 5} c_1 x^5 + \frac{1}{2\cdot 4\cdot 6} c_0 x^6$$

$$+ \frac{1}{3\cdot 5\cdot 7} c_1 x^7 + \frac{1}{2\cdot 4\cdot 6\cdot 8} c_0 x^8 + \cdots$$

$$= c_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{2\cdot 4} x^4 + \frac{1}{2\cdot 4\cdot 6} x^6 + \frac{1}{2\cdot 4\cdot 6\cdot 8} x^8 + \cdots \right]$$

$$+ c_1 \left[x + \frac{1}{3} x^3 + \frac{1}{3\cdot 5} x^5 + \frac{1}{3\cdot 5\cdot 7} x^7 + \cdots \right]$$

NOTE THAT

$$\frac{1}{2\cdot4\cdot6\cdot8} = \frac{1}{(2\cdot1)(2\cdot2)(2\cdot3)(2\cdot4)} = \frac{1}{2^4\cdot4!}$$

$$\frac{1}{1\cdot3\cdot5\cdot7} = \frac{2\cdot4\cdot6}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} = \frac{2^3\cdot3!}{7!}$$

AND SO

$$y = c_0\left[\frac{1}{2^0\cdot0!}x^{2\cdot0} + \frac{1}{2^1\cdot1!}x^{2\cdot1} + \frac{1}{2^2\cdot2!}x^{2\cdot2} + \frac{1}{2^3\cdot3!}x^{2\cdot3} + \frac{1}{2^4\cdot4!}x^{2\cdot4} + \cdots\right]$$

$$+ c_1\left[\frac{2^0\cdot0!}{1!}x^1 + \frac{2^1\cdot1!}{3!}x^3 + \frac{2^2\cdot2!}{5!}x^5 + \frac{2^3\cdot3!}{7!}x^7 + \cdots\right]$$

$$= c_0\sum_{k=0}^{\infty}\frac{1}{2^k\cdot k!}x^{2k} + c_1\sum_{k=0}^{\infty}\frac{2^k\cdot k!}{(2k+1)!}x^{2k+1}.$$

THERE IS NOTHING TO DETERMINE c_0 AND c_1 SO THEY ARE ARBITRARY.

<u>NOTE</u>: THIS TIME THE FUNCTIONS $\sum_{k=0}^{\infty}\frac{1}{2^k k!}x^{2k}$ AND $\sum_{k=0}^{\infty}\frac{2^k k!}{(2k+1)!}x^{2k+1}$ CANNOT BE IDENTIFIED WITH ANY SIMPLE COMBINATION OF ELEMENTARY FUNCTIONS, BUT THEY ARE, NEVERTHELESS, PERFECTLY FINE SOLUTIONS TO THE DIFFERENTIAL EQUATION.

EXERCISES : FIND SERIES SOLUTIONS TO EACH OF THE FOLLOWING DIFFERENTIAL EQUATIONS. WRITE THE ANSWERS IN Σ-NOTATION.

1. $y' + y = 0$
2. $y'' + 4y = 0$
3. $(x-1)y' + 2y = 0$
4. $y'' + 2xy' + 2y = 0$

SOLUTIONS :

1. $y' + y = 0$: $\quad y = \sum_{k=0}^{\infty} c_k x^k$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+1) c_{n+1} + c_n] x^n = 0 \quad \Rightarrow$$

$$(n+1)c_{n+1} + c_n = 0 \qquad , n \geq 0$$

$$c_{n+1} = -\frac{1}{n+1}c_n \qquad , n \geq 0$$

$n=0$: $\quad c_1 = -\frac{1}{1}c_0$

$n=1$: $\quad c_2 = -\frac{1}{2}c_1 = \frac{1}{1\cdot 2}c_0$

$n=2$: $\quad c_3 = -\frac{1}{3}c_2 = -\frac{1}{1\cdot 2\cdot 3}c_0$

$n=3$: $\quad c_4 = -\frac{1}{4}c_3 = \frac{1}{1\cdot 2\cdot 3\cdot 4}c_0$

$\vdots$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$= c_0 - \frac{1}{1}c_0 x + \frac{1}{1\cdot 2}c_0 x^2 - \frac{1}{1\cdot 2\cdot 3}c_0 x^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}c_0 x^4 - \cdots$$

$$= c_0\left[\frac{1}{0!} - \frac{1}{1!}x^1 + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}x^k$$

<u>NOTE</u>: THIS IS $c_0 e^{-x}$

2. $\quad y'' + 4y = 0$:

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + 4c_n]x^n = 0 \implies$$

$$(n+2)(n+1)c_{n+2} + 4c_n = 0 \quad , n \geq 0$$

$$c_{n+2} = -\frac{4}{(n+2)(n+1)} c_n \quad , n \geq 0$$

$n=0$: $\quad c_2 = -\frac{4}{2\cdot 1} c_0$

$n=1$: $\quad c_3 = -\frac{4}{3\cdot 2} c_1$

$n=2$: $\quad c_4 = -\frac{4}{4\cdot 3} c_2 = \frac{4^2}{4\cdot 3\cdot 2\cdot 1} c_0$

$n=3$: $\quad c_5 = -\frac{4}{5\cdot 4} c_3 = \frac{4^2}{5\cdot 4\cdot 3\cdot 2} c_1$

$\vdots$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x - \frac{4}{2!} c_0 x^2 - \frac{4}{3!} c_1 x^3 + \frac{4^2}{4!} c_0 x^4 + \frac{4^2}{5!} c_1 x^5 + \cdots$$

$$= c_0 \left[1 - \frac{4}{2!} x^2 + \frac{4^2}{4!} x^4 - \cdots \right]$$

$$+ c_1 \left[x - \frac{4}{3!} x^3 + \frac{4^2}{5!} x^5 - \cdots \right]$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k+1)!} x^{2k+1}$$

$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2x)^{2k} + \frac{c_1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2x)^{2k+1}$$

<u>NOTE</u> : THIS IS $c_0 \cos(2x) + \frac{c_1}{2} \sin(2x)$

3. $(x-1)y' + 2y = 0 \quad : \quad y = \sum_{k=0}^{\infty} c_k x^k$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$\begin{aligned}
(x-1)y' &= (x-1)\sum_{k=1}^{\infty} k c_k x^{k-1} \\
&= x\sum_{k=1}^{\infty} k c_k x^{k-1} - \sum_{k=1}^{\infty} k c_k x^{k-1} \\
&= \sum_{k=1}^{\infty} k c_k x^{k} - \sum_{k=1}^{\infty} k c_k x^{k-1} \\
&= \sum_{n=1}^{\infty} n c_n x^{n} - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n}
\end{aligned}$$

$$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} n c_n x^n - (0+1) c_{0+1} x^0 - \sum_{n=1}^{\infty} (n+1) c_{n+1} x^n + 2c_0 x^0$$
$$+ \sum_{n=1}^{\infty} 2c_n x^n = 0$$

$$(-c_1 + 2c_0) + \sum_{n=1}^{\infty} [n c_n - (n+1) c_{n+1} + 2c_n] x^n = 0$$

$$(-c_1 + 2c_0) + \sum_{n=1}^{\infty} [(n+2) c_n - (n+1) c_{n+1}] x^n = 0$$

$\Rightarrow$

$$c_1 = 2c_0$$

$$c_{n+1} = \frac{n+2}{n+1} c_n \quad , n \geq 1$$

$n = 1$: $\quad c_2 = \frac{3}{2} c_1 = (2) \frac{3}{2} c_0$

$n = 2$: $\quad c_3 = \frac{4}{3} c_2 = (2) \frac{4}{3} \cdot \frac{3}{2} c_0$

$n = 3$: $\quad c_4 = \frac{5}{4} c_3 = (2) \frac{5}{4} \cdot \frac{4}{3} \cdot \frac{3}{2} c_0$

$n = 4$: $\quad c_5 = \frac{6}{5} c_4 = (2) \frac{6}{5} \cdot \frac{5}{4} \cdot \frac{4}{3} \cdot \frac{3}{2} c_0$

CANCELLING GIVES

$$c_1 = 2c_0$$
$$c_2 = 3c_0$$
$$c_3 = 4c_0$$
$$c_4 = 5c_0$$
$$c_5 = 6c_0$$

SO

$$y = c_0 [1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots]$$
$$= c_0 \sum_{k=0}^{\infty} (k+1) x^k$$

4. $y'' + 2xy' + 2y = 0$

$$y = \sum_{k=0}^{\infty} c_k x^k$$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1} \quad \text{AND} \quad 2xy' = \sum_{k=1}^{\infty} 2k c_k x^k$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$(0+2)(0+1) c_{0+2} x^0 + 2c_0 x^0 +$$

$$\sum_{n=1}^{\infty} [(n+2)(n+1) c_{n+2} + 2(n+1) c_n] x^n = 0$$

$$\Rightarrow \qquad 2c_2 + 2c_0 = 0$$

$$(n+2)(n+1) c_{n+2} + 2(n+1) c_n = 0 \quad , n \geq 1$$

$$\Rightarrow \qquad c_2 = - c_0$$

$$\Rightarrow \qquad c_{n+2} = - \frac{2}{n+2} c_n \quad , n \geq 0$$

$n = 1$: $\quad c_3 = - \frac{2}{3} c_1$

$n = 2$: $\quad c_4 = - \frac{2}{4} c_2 = - \frac{2}{4} (- c_0) = \frac{(-2)^1}{4} c_0$

$n = 3$: $\quad c_5 = - \frac{2}{5} c_3 = - \frac{2}{5} \left(- \frac{2}{3} c_1\right) = \frac{(-2)^2}{3 \cdot 5} c_1$

$$n = 4 : \quad c_6 = -\frac{2}{6} c_4 = -\frac{2}{6}\left(\frac{(-2)^1}{4} c_0\right) = \frac{(-2)^2}{4 \cdot 6} c_0$$

$$n = 5 : \quad c_7 = -\frac{2}{7} c_5 = -\frac{2}{7}\left(\frac{(-2)^2}{3 \cdot 5} c_1\right) = \frac{(-2)^3}{3 \cdot 5 \cdot 7} c_1$$

$$n = 6 : \quad c_8 = -\frac{2}{8} c_6 = -\frac{2}{8}\left(\frac{(-2)^2}{4 \cdot 6} c_0\right) = \frac{(-2)^3}{4 \cdot 6 \cdot 8} c_0$$

EXCEPT FOR THE FACTORS OF -2 THE PATTERN HERE IS EXACTLY THE SAME AS IN EXAMPLE 3, PAGE 8.

$$y = c_0 \left[1 + \frac{(-2)^1}{2^1 \cdot 1!} x^2 + \frac{(-2)^2}{2^2 \cdot 2!} x^4 + \frac{(-2)^3}{2^3 \cdot 3!} x^6 + \frac{(-2)^4}{2^4 \cdot 4!} x^8 + \cdots \right]$$

$$+ c_1 \left[\frac{(-2)^0 2^0 \cdot 0!}{1!} x^1 + \frac{(-2)^1 2^1 \cdot 1!}{3!} x^3 + \frac{(-2)^2 \cdot 2^2 \cdot 2!}{5!} x^5 + \frac{(-2)^3 \cdot 2^3 \cdot 3!}{7!} x^7 + \cdots \right]$$

$$= c_0 \left[1 + \frac{(-1)^1 \cdot 2}{2^1 \cdot 1!} x^2 + \frac{(-1)^2 \cdot 2^2}{2^2 \cdot 2!} x^4 + \frac{(-1)^3 2^3}{2^3 \cdot 3!} x^6 + \frac{(-1)^4 \cdot 2^4}{2^4 \cdot 4!} x^8 + \cdots \right]$$

$$+ c_1 \left[\frac{(-1)^0 \cdot 2^0 \cdot 2^0 \cdot 0!}{1!} x^1 + \frac{(-1)^1 \cdot 2^1 \cdot 2^1 \cdot 1!}{3!} x^3 + \frac{(-1)^2 \cdot 2^2 \cdot 2^2 \cdot 2!}{5!} x^5 + \frac{(-1)^3 \cdot 2^3 \cdot 2^3 \cdot 3!}{7!} x^7 + \cdots \right]$$

$$= c_0 \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} \right]$$

$$+ c_1 \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2^{2k} \cdot k!}{(2k+1)!} x^{2k+1}$$