SERIES OF NON-NEGATIVE TERMS

FOR SERIES WITH NO NEGATIVE TERMS WE DESCRIBE FOUR MORE "TESTS":

1. COMPARISON
2. LIMIT COMPARISON
3. RATIO
4. ROOT

1. MOTIVATION:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{CONVERGES} \quad (p\text{-SERIES}, \ p=2)$$

NOW CONSIDER $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$.

$$\frac{1}{k^2+1} < \frac{1}{k^2} \quad \Rightarrow$$

$$\frac{1}{1^2+1} + \frac{1}{2^2+1} + \cdots + \frac{1}{n^2+1} < \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

$$< \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

THUS, THE PARTIAL SUMS OF $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ ARE INCREASING (POSITIVE TERMS) AND BOUNDED FROM ABOVE (BY $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$) SO $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ CONVERGES.

FOR SIMPLICITY WE WILL WRITE ALL OF THIS AS

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1} \ll \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

WHICH CONVERGES

IS TERM-BY-TERM
LESS THAN
(EVENTUALLY)

ON THE OTHER HAND,

$$\sum_{k=1}^{\infty} \frac{1}{k} \quad \text{DIVERGES} \qquad \text{(HARMONIC SERIES)}$$

AND $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}}$ IS TERM-BY-TERM GREATER THAN $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ SO ITS PARTIAL SUMS ARE LARGER THAN THOSE OF $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, WHICH BLOW UP. THUS, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}}$ DIVERGES AS WELL.

SYMBOLICALLY,

$$\sum_{k=1}^{\infty} \frac{1}{k-\frac{1}{2}} \gg \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

WHICH DIVERGES

IS TERM-BY-TERM
GREATER THAN
(EVENTUALLY)

HERE IS THE GENERAL RESULT.

<u>THEOREM</u> (<u>COMPARISON</u> <u>TEST</u>) : LET $\sum_{k=1}^{\infty} a_k$ AND $\sum_{k=1}^{\infty} b_k$ BE TWO SERIES OF NON-NEGATIVE TERMS AND SUPPOSE THAT (EVENTUALLY)

$$a_k \le b_k .$$

THEN

(1) $\sum_{k=1}^{\infty} b_k$ CONVERGES $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ CONVERGES

(2) $\sum_{k=1}^{\infty} a_k$ DIVERGES $\Rightarrow$ $\sum_{k=1}^{\infty} b_k$ DIVERGES

<u>EXAMPLES</u> :

1. $$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

INTUITION SAYS THIS SHOULD CONVERGE (BECAUSE $k!$ GROWS SO QUICKLY). TO SHOW THIS WE MUST FIND A SERIES WHICH

(1) WE KNOW CONVERGES, AND

(2) IS (EVENTUALLY) TERM-BY-TERM GREATER THAN OR EQUAL TO $\sum_{k=1}^{\infty} \frac{1}{k!}$

$$\frac{1}{k!} \le \frac{1}{2^k} \quad \text{FOR ALL } k \geq 4$$

THUS,

$$\sum_{k=0}^{\infty} \frac{1}{k!} << \sum_{k=0}^{\infty} \frac{1}{2^k} < \infty$$ (GEOMETRIC SERIES WITH $r = \frac{1}{2}$)

SO $\sum_{k=0}^{\infty} \frac{1}{k!}$ CONVERGES.

2. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

LOOKS LIKE IT SHOULD DIVERGE (LARGER THAN HARMONIC SERIES)

FOR $k \geq 3$,

$$\frac{\ln k}{k} \geq \frac{1}{k}$$

($\ln 1 = 0$ AND $\ln 2 = 0.69$, BUT $\ln 3 = 1.09$) SO

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} >> \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$ (HARMONIC SERIES)

SO $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ DIVERGES.

THE COMPARISON TEST CAN BE DIFFICULT TO USE BECAUSE YOU NEED TO "KNOW" THE ANSWER (CONVERGE OR DIVERGE) AND THEN FIND A SERIES TO COMPARE TO IN ORDER TO PROVE IT.

IT ALSO FAILS TO BE USEFUL FOR SERIES LIKE $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$ WHERE THE ANSWER SEEMS OBVIOUS BUT THE NATURAL INEQUALITIES GO

THE WRONG WAY : $\sum_{k=2}^{\infty} \frac{1}{k^2-1} >> \sum_{k=2}^{\infty} \frac{1}{k^2}$ IMPLIES NOTHING

THE FOLLOWING TEST, WHICH SAYS THAT IT IS REALLY ONLY NECESSARY TO COMPARE a_k AND b_k " IN THE LIMIT $k \to \infty$ " CAN BE A SIMPLER ALTERNATIVE.

2. <u>THEOREM</u> (<u>LIMIT COMPARISON TEST</u>): LET $\sum_{k=1}^{\infty} a_k$ AND $\sum_{k=1}^{\infty} b_k$ BE TWO SERIES OF POSITIVE TERMS AND SUPPOSE THAT

$$\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$$

IS FINITE AND NONZERO (NEITHER a_k NOR b_k DOMINATES THE OTHER). THEN THE SERIES EITHER BOTH CONVERGE OR BOTH DIVERGE.

<u>EXAMPLES</u> :

1. $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$

WE CERTAINLY EXPECT THIS TO CONVERGE (" BASICALLY THE SAME AS $\sum_{k=2}^{\infty} \frac{1}{k^2}$ "), BUT AS WE NOTED ABOVE, THE SIMPLE COMPARISON WITH $\sum_{k=2}^{\infty} \frac{1}{k^2}$ GOES THE WRONG WAY.

HOWEVER,

$$\lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1$$

WHICH IS FINITE AND NONZERO SO

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \text{ CONVERGES} \Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^2-1} \text{ CONVERGES}$$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$

<u>NOTE</u>: ALTHOUGH IT'S EASY TO GUESS THAT THIS SHOULD CONVERGE (" BASICALLY THE SAME AS $\frac{3k^3}{k^7} = \frac{3}{k^4}$ ") WE DON'T EVEN NEED TO BE THAT CLEVER. WE JUST NEED TO DIVIDE BY SOMETHING THAT MAKES THE LIMIT COME OUT FINITE AND NONZERO AND THIS JUST MEANS MAKING THE DEGREES THE SAME IN THE NUMERATOR AND DENOMINATOR.

$$\lim_{k\to\infty} \frac{\frac{3k^3-2k^2+4}{k^7-k^3+2}}{\frac{1}{k^4}} = \lim_{k\to\infty} \frac{3k^7-2k^6+4k^4}{k^7-k^3+2} = 3$$

WHICH IS FINITE AND NONZERO SO

$$\sum_{k=1}^{\infty} \frac{1}{k^4} \text{ CONVERGES} \Rightarrow \sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2} \text{ CONVERGES}$$

3. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

SINCE $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, WE ALSO HAVE $\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$

WHICH IS FINITE AND NONZERO SO

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ DIVERGES } \Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \text{ DIVERGES.}$$

THE NEXT TEST IS GENERALLY EVEN EASIER TO USE AND WILL BE THE MOST IMPORTANT TEST WHEN WE GET AROUND TO TURNING "TAYLOR POLYNOMIALS" INTO "TAYLOR SERIES"

3. <u>THEOREM</u> (<u>RATIO</u> <u>TEST</u>): LET $\sum_{k=1}^{\infty} a_k$ BE A SERIES OF POSITIVE TERMS AND LET

$$\rho = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}.$$

THEN

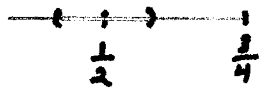
1. IF $\rho < 1$, THE SERIES CONVERGES.
2. IF $\rho > 1$ (INCLUDING $\rho = \infty$), THE SERIES DIVERGES.
3. IF $\rho = 1$, THE TEST FAILS.

PROOF OF THIS IS A BIT TRICKY, BUT WE CAN GET SOME CLUE AS TO WHY IT'S TRUE. SUPPOSE, E.G.,

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{2}.$$

THEN, EVENTUALLY,

$$\frac{a_{k+1}}{a_k} < \frac{3}{4}$$



$$a_{k+1} < \frac{3}{4} a_k$$

IGNORE THE FIRST FEW TERMS AND SUPPOSE THIS IS TRUE FOR ALL k.

THEN

$$a_2 < \frac{3}{4} a_1$$

$$a_3 < \frac{3}{4} a_2 < \left(\frac{3}{4}\right)^2 a_1$$

$$a_4 < \frac{3}{4} a_3 < \left(\frac{3}{4}\right)^3 a_1$$

$$\vdots$$

SO

$$\sum a_k << \sum a_1 \left(\frac{3}{4}\right)^{k-1} < \infty$$

(GEOMETRIC SERIES)

AND $\sum a_k$ CONVERGES BY THE COMPARISON TEST.

WHEN IT WORKS, THE RATIO TEST IS REALLY EASY TO USE.

<u>EXAMPLES</u> :

1. $\displaystyle\sum_{k=0}^{\infty} \frac{10^k}{k!}$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}} = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k}$$

$$= \lim_{k\to\infty} \frac{10}{k+1} = 0 < 1 \quad \text{SO}$$

$\displaystyle\sum_{k=0}^{\infty} \frac{10^k}{k!}$ CONVERGES.

2. $\displaystyle\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$

$$\lim_{k\to\infty} \frac{(2(k+1))!}{4^{k+1}} \cdot \frac{4^k}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)!}{4^{k+1}} \cdot \frac{4^k}{(2k)!}$$

$$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{4} = \infty$$

SO $\displaystyle\sum_{k=3}^{\infty} \frac{(2k)!}{4^k}$ DIVERGES.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}$

$$\lim_{k\to\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} = \lim_{k\to\infty} \frac{2k-1}{2k+1} = 1$$

SO HERE, UNFORTUNATELY, THE RATIO TEST DOESN'T WORK.

<u>QUESTION</u>: HOW WOULD YOU TEST $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-1}$?

IS IT GOING TO CONVERGE OR DIVERGE ?

THE FINAL TEST IN THIS SECTION DOESN'T COME UP THAT OFTEN, BUT, WHEN IT DOES, IT'S TENDS TO BE THE ONLY ONE THAT WORKS WITHOUT A LOT OF EFFORT.

4. <u>THEOREM</u> (<u>ROOT</u> <u>TEST</u>) : LET $\sum_{k=1}^{\infty} a_k$ BE A SERIES OF POSITIVE TERMS AND LET

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k} \,.$$

THEN

1. IF $\rho < 1$, THE SERIES CONVERGES.
2. IF $\rho > 1$ (INCLUDING $\rho = \infty$), THE SERIES DIVERGES.
3. IF $\rho = 1$, THE TEST FAILS.

THIS IS PROVED IN MUCH THE SAME WAY AS THE RATIO TEST.

<u>EXAMPLES</u>:

1. $\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$: $\lim_{k\to\infty} \sqrt[k]{\frac{1}{(\ln(k+1))^k}} = \lim_{k\to\infty} \frac{1}{\ln(k+1)}$

$= 0 < 1$

SO THE SERIES CONVERGES.

2. $\sum_{k=1}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$: $\lim_{k\to\infty} \sqrt[k]{\left(\frac{4k-5}{2k+1}\right)^k} = \lim_{k\to\infty} \frac{4k-5}{2k+1}$

$= 2 > 1$

SO THE SERIES DIVERGES.