SMOOTH MAPPINGS ON MANIFOLDS

X = DIFFERENTIABLE MANIFOLD OF DIMENSION n.

$$f : X \to \mathbb{R}$$

A CONTINUOUS REAL-VALUED FUNCTION ON X.

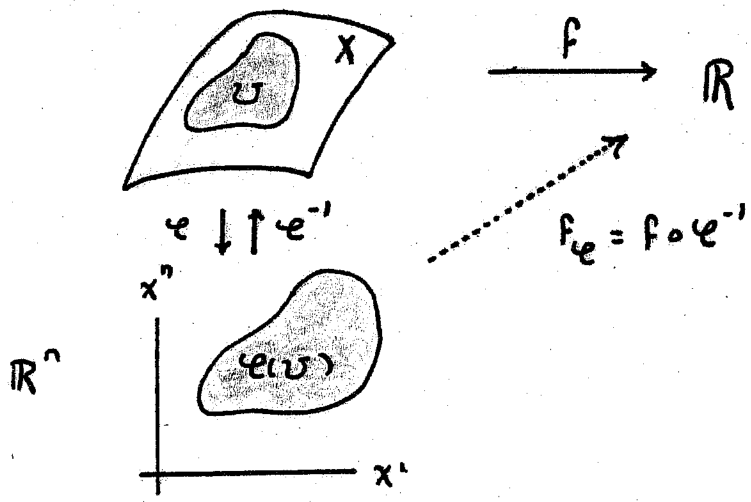
(U, φ) A CHART ON X WITH COORDINATE FUNCTIONS

$$x^i = \pi^i \circ \varphi : U \to \mathbb{R}$$

(SO, FOR EACH $p \in U$, $\varphi(p) = (x^1(p), \ldots, x^n(p))$).

COMPOSING f WITH φ^{-1} WE OBTAIN A <u>LOCAL COORDINATE EXPRESSION</u> FOR f:

$$f_\varphi = f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$$



$$f_\varphi(x^1, \ldots, x^n) = (f \circ \varphi^{-1})(x^1, \ldots, x^n)$$

MAY OR MAY NOT BE SMOOTH.

EXAMPLES: LET $X = S^2 = \{ p = (p^1, p^2, p^3) \in \mathbb{R}^3 : \|p\|^2 = 1 \}$ AND DEFINE THE HEIGHT FUNCTION h ON S^2 BY

$$h(p) = h(p^1, p^2, p^3) = p^3 .$$

WE FIND LOCAL COORDINATE EXPRESSIONS FOR h RELATIVE TO A FEW CHARTS ON S^2.

1. (U, φ) $\quad U = \{ p = (p^1, p^2, p^3) \in S^2 : p^3 > 0 \}$

$$\varphi(p) = \varphi(p^1, p^2, p^3) = (p^1, p^2)$$

$$x^1 = \pi^1 \circ \varphi \qquad x^1(p) = p^1$$

$$x^2 = \pi^2 \circ \varphi \qquad x^2(p) = p^2$$

$$\varphi(U) = \{ (x^1, x^2) \in \mathbb{R}^2 : (x^1)^2 + (x^2)^2 < 1 \}$$

$$\varphi^{-1} : \varphi(U) \to U$$

$$\varphi^{-1}(x^1, x^2) = (x^1, x^2, \sqrt{1 - (x^1)^2 - (x^2)^2})$$

$$\begin{aligned} h_\varphi(x^1, x^2) &= h(\varphi^{-1}(x^1, x^2)) \\ &= h(x^1, x^2, \sqrt{1-(x^1)^2-(x^2)^2}) \\ &= \sqrt{1-(x^1)^2-(x^2)^2} \end{aligned}$$

2. (V, ψ) $\quad V = \{ p = (p^1, p^2, p^3) \in S^2 : p^1 > 0 \}$

$$\psi(p) = \psi(p^1, p^2, p^3) = (p^2, p^3)$$

$$y^1 = \pi^1 \circ \psi \qquad y^1(p) = p^2$$

$$y^2 = \pi^2 \circ \psi \qquad y^2(p) = p^3$$

$$\psi(V) = \{ (y^1, y^2) \in \mathbb{R}^2 : (y^1)^2 + (y^2)^2 < 1 \}$$

$$\psi^{-1} : \psi(V) \to V$$

$$\psi^{-1}(y^1, y^2) = \left(\sqrt{1-(y^1)^2-(y^2)^2},\ y^1, y^2\right)$$

$$h_\psi(y^1, y^2) = h(\psi^{-1}(y^1, y^2)) = h\left(\sqrt{1-(y^1)^2-(y^2)^2},\ y^1, y^2\right)$$

$$= y^2$$

3. (U_S, φ_S)

$$U_S = S^2 - \{(0,0,1)\}$$

$$\varphi_S(p) = \varphi_S(p^1, p^2, p^3) = \left(\frac{p^1}{1-p^3}, \frac{p^2}{1-p^3}\right)$$

$$x^1 = \pi^1 \circ \varphi_S \qquad x^1(p) = \frac{p^1}{1-p^3}$$

$$x^2 = \pi^2 \circ \varphi_S \qquad x^2(p) = \frac{p^2}{1-p^3}$$

$$\varphi_S(U_S) = \mathbb{R}^2$$

$$\varphi_S^{-1} : \varphi_S(U_S) \to U_S$$

$$\varphi_S^{-1}(x^1, x^2) = \left(\frac{2x^1}{(x^1)^2+(x^2)^2+1},\ \frac{2x^2}{(x^1)^2+(x^2)^2+1},\ \frac{(x^1)^2+(x^2)^2-1}{(x^1)^2+(x^2)^2+1}\right)$$

$$h_{\varphi_S}(x^1, x^2) = h(\varphi_S^{-1}(x^1, x^2))$$

$$= \frac{(x^1)^2+(x^2)^2-1}{(x^1)^2+(x^2)^2+1}$$

THUS, THE LOCAL COORDINATE EXPRESSIONS FOR A GIVEN REAL-VALUED FUNCTION CAN LOOK QUITE DIFFERENT. HOWEVER,

IF (U,φ) AND (V,ψ) ARE C^∞-RELATED, THEN ON $U \cap V$,

$$f_\varphi = f\circ\varphi^{-1} = (f\circ\psi^{-1})\circ(\psi\circ\varphi^{-1}) = f_\psi\circ(\psi\circ\varphi^{-1})$$

$$f_\psi = f\circ\psi^{-1} = (f\circ\varphi^{-1})\circ(\varphi\circ\psi^{-1}) = f_\varphi\circ(\varphi\circ\psi^{-1})$$

SO f_φ IS SMOOTH IF AND ONLY IF f_ψ IS SMOOTH.

THIS IS THE REASON THE FOLLOWING DEFINITION MAKES SENSE :

A REAL-VALUED FUNCTION $f : X \to \mathbb{R}$ ON A DIFFERENTIABLE MANIFOLD X IS SAID TO BE <u>SMOOTH</u> (OR <u>C</u>$^\infty$) IF ITS COORDINATE EXPRESSIONS

$$f_\varphi = f\circ\varphi^{-1} : \varphi(U) \to \mathbb{R}$$

ARE SMOOTH FOR ALL CHARTS (U,φ) IN SOME (ATLAS FOR) THE DIFFERENTIABLE STRUCTURE FOR X.

<u>EXAMPLE</u> : THE HEIGHT FUNCTION ON S^2 IS SMOOTH.

IF W IS AN OPEN SUBSET OF X AND

$$f : W \to \mathbb{R}$$

IS A REAL-VALUED FUNCTION ON W, THEN f IS SAID TO BE <u>SMOOTH</u>, OR <u>C</u>$^\infty$, IF IT IS SMOOTH AS A FUNCTION ON THE OPEN SUBMANIFOLD W OF X.

IF THE DOMAIN ISN'T OPEN WE DEFINE SMOOTHNESS "POINTWISE":

LET A BE A SUBSET OF X AND

$$f : A \to \mathbb{R}$$

A CONTINUOUS FUNCTION (A IS A TOPOLOGICAL SUBSPACE OF X). IF $p \in A$ WE SAY THAT f IS <u>SMOOTH</u>, OR <u>C^∞</u>, <u>AT</u> p IF $\exists$ OPEN NEIGHBORHOOD W OF p CONTAINED IN A SUCH THAT $f|W$ IS SMOOTH.

> <u>NOTE</u>: IF f IS SMOOTH AT EVERY p IN A, THEN A IS NECESSARILY OPEN.

NOTICE THAT THIS DEFINITION OF SMOOTHNESS FOR REAL-VALUED FUNCTIONS DEPENDS ENTIRELY ON THE CHOICE OF DIFFERENTIABLE STRUCTURE (ALTHOUGH WE WILL EVENTUALLY SEE THAT TWO DIFFERENT DIFFERENTIABLE STRUCTURES ON THE SAME SPACE X CAN DETERMINE EXACTLY THE SAME SMOOTH FUNCTIONS).

<u>EXERCISE</u> 69: LET $f : X \to \mathbb{R}$ AND $g : X \to \mathbb{R}$ BE SMOOTH. SHOW THAT $f+g$ AND fg (DEFINED BY $(f+g)(p) = f(p) + g(p)$ AND $(fg)(p) = f(p)g(p)$) ARE SMOOTH. THE SET $C^\infty(X)$ OF ALL SMOOTH REAL-VALUED FUNCTIONS ON X IS THEREFORE A <u>RING</u>, AND ALSO A VECTOR SPACE IF SCALAR MULTIPLICATION IS IDENTIFIED WITH MULTIPLICATION BY CONSTANT FUNCTIONS. THUS, $C^\infty(X)$ IS AN <u>ALGEBRA</u>.

THE NEXT RESULT IS A USEFUL TECHNICAL TOOL THAT I DO NOT WISH TO TAKE THE TIME TO PROVE NOW (BECAUSE THE PROOF IS RATHER LONG AND IS REALLY ANALYSIS RATHER THAN TOPOLOGY OR GEOMETRY).

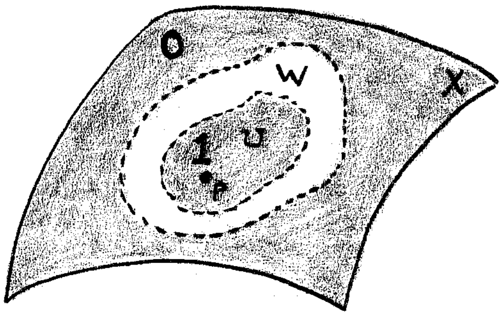
<u>LEMMA</u>: LET X BE A SMOOTH MANIFOLD, W AN OPEN SET IN X AND $p_0 \in W$. THEN $\exists$ OPEN NEIGHBORHOOD U OF p_0 CONTAINED IN W AND A $g \in C^\infty(X)$ SATISFYING

$$0 \le g(p) \le 1 \qquad \forall p \in X$$

$$g(p) = 1 \qquad \forall p \in U$$

$$g(p) = 0 \qquad \forall p \in X - W$$

(g IS CALLED A <u>BUMP FUNCTION</u> AT p IN W).



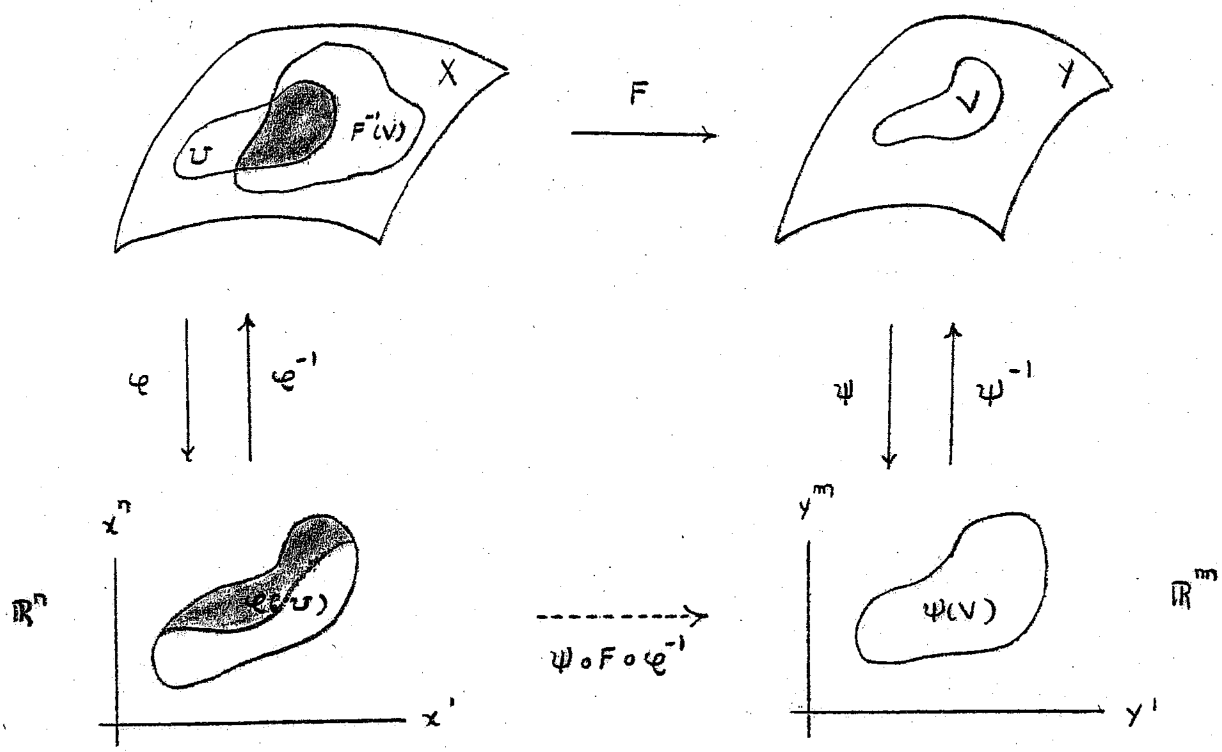
THUS FAR WE HAVE DISCUSSED ONLY SMOOTH <u>REAL-VALUED</u> FUNCTIONS ON A MANIFOLD. NOW WE TURN TO THE SMOOTHNESS OF MAPS BETWEEN ANY TWO MANIFOLDS.

LET X AND Y BE SMOOTH MANIFOLDS OF DIMENSION n AND m, RESPECTIVELY, AND LET

$$F : X \to Y$$

BE A CONTINUOUS MAP. LET (U, φ) BE A CHART FOR X AND (V, ψ) A CHART FOR Y WITH

$$U \cap F^{-1}(V) \neq \emptyset$$



THE <u>COORDINATE</u> <u>EXPRESSION</u> FOR F RELATIVE TO (U, φ) AND (V, ψ) IS THE MAP

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$$

AND F IS SAID TO BE <u>SMOOTH</u> (OR <u>C^∞</u>) IF ITS COORDINATE EXPRESSIONS ARE SMOOTH FOR ALL CHARTS (U, φ) IN SOME ATLAS FOR X AND ALL CHARTS (V, ψ) IN SOME ATLAS FOR Y WITH $U \cap F^{-1}(V) \neq \emptyset$ FOR SOME (U, φ).

EXERCISE 70 : PROVE EACH OF THE FOLLOWING PROPERTIES OF SMOOTH MAPS.

1. SMOOTHNESS IS A LOCAL PROPERTY, I.E., $F : X \to Y$ IS SMOOTH IFF $\forall\ p \in X\ \exists$ OPEN NBD W OF p IN X S.T. $F|W$ IS SMOOTH (AS A MAP ON THE OPEN SUBMANIFOLD W OF X).

2. IF $F : X \to Y$ AND $G : Y \to Z$ ARE SMOOTH, THEN $G \circ F : X \to Z$ IS SMOOTH.

3. IF $F : X \to Y$ IS SMOOTH AND X' IS A SUBMANIFOLD OF X, THEN $F|X' : X' \to Y$ IS SMOOTH.

4. IF $F : X \to Y$ IS SMOOTH AND Y' IS A SUBMANIFOLD OF Y WITH $F(X) \subseteq Y'$, THEN $F : X \to Y'$ IS SMOOTH.

EXAMPLES :

1. (THE HOPF MAP) WE HAVE ALREADY CONSTRUCTED A CONTINUOUS MAP

$$H : S^3 \to S^2$$

OF $S^3 = \{ (z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1 \}$ ONTO $S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$ GIVEN BY

$$H(z^1, z^2) = (2 \operatorname{Re}(\bar{z}^1 z^2),\ 2 \operatorname{Im}(\bar{z}^1 z^2),\ |z^2|^2 - |z^1|^2)$$

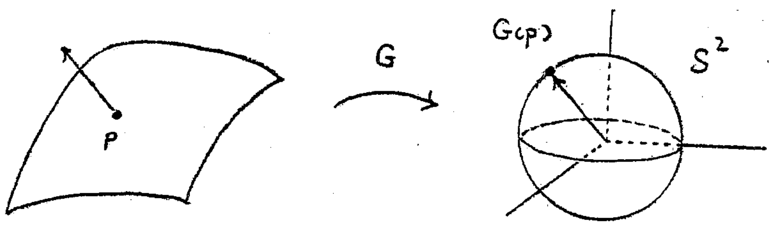
FORGET ABOUT S^3 AND S^2 FOR A MOMENT AND CONSIDER THE MAP FROM $\mathbb{R}^4 = \mathbb{C}^2$ TO $\mathbb{R}^3$ GIVEN BY

$$(z^1, z^2) \in \mathbb{R}^4 \longrightarrow (2\,\mathrm{Re}\,(\bar{z}^1 z^2),\ 2\,\mathrm{Im}\,(\bar{z}^1 z^2),\ |z^2|^2 - |z^1|^2)$$

WRITING THIS OUT IN TERMS OF REAL AND IMAGINARY PARTS $z^1 = (x^1, x^2)$, $z^2 = (x^3, x^4)$, THE COMPONENT FUNCTIONS ARE JUST POLYNOMIALS SO THE MAP IS SMOOTH. BY EXERCISE 70 (3) ITS RESTRICTION TO THE SUBMANIFOLD S^3 IS SMOOTH. THIS RESTRICTION MAPS INTO S^2 AND IS JUST THE HOPF MAP SO, BY EXERCISE 70 (4), THE HOPF MAP IS SMOOTH.

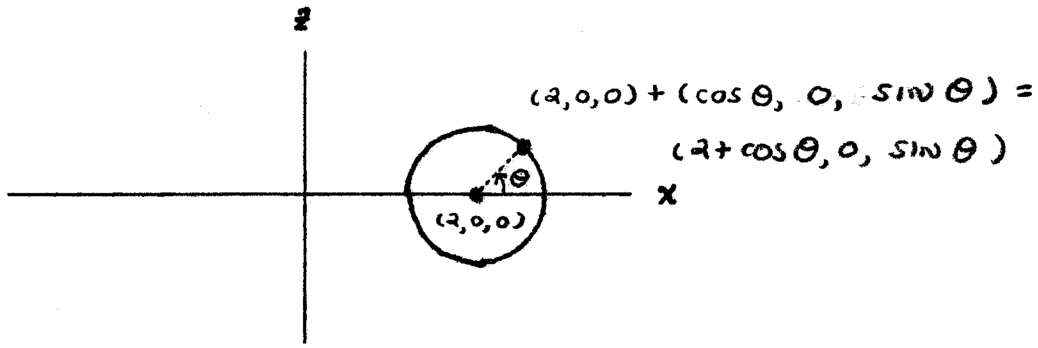
2. (THE GAUSS MAP OF THE TORUS)

AN IMPORTANT TOOL IN CLASSICAL DIFFERENTIAL GEOMETRY IS THE "GAUSS MAP" OF A SURFACE. IT IS A MAP FROM THE SURFACE TO S^2 WHICH ESSENTIALLY SENDS EACH POINT TO THE UNIT NORMAL VECTOR TO THE SURFACE AT THAT POINT, THOUGHT OF AS AN ELEMENT OF S^2.

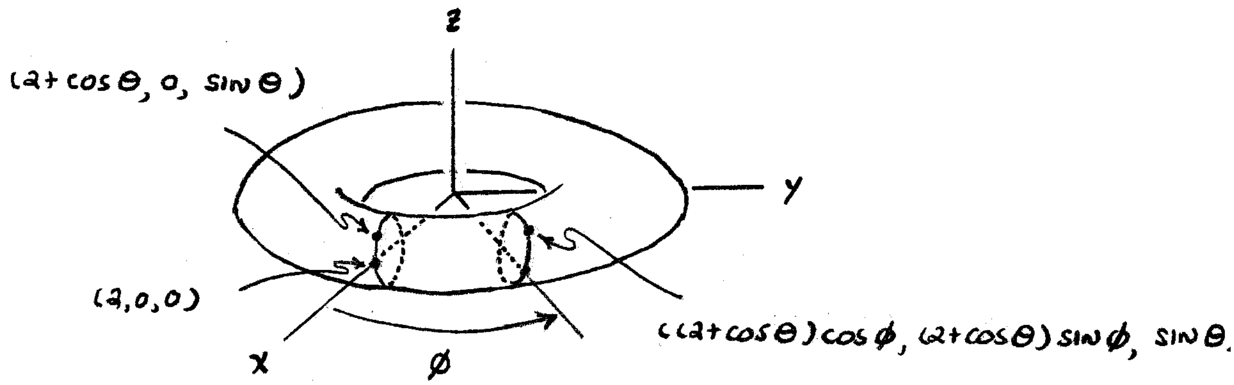


FOR EXAMPLE, THE GAUSS MAP OF S^2 IS THE IDENTITY MAP ON S^2 AND THE GAUSS MAP OF A PLANE IS A CONSTANT MAP TO S^2. WE WILL WRITE OUT THE GAUSS MAP FOR A TORUS AND SHOW THAT IT IS A SMOOTH MAP.

WE WILL VIEW THE TORUS AS A SURFACE OF REVOLUTION IN $\mathbb{R}^3$. SPECIFICALLY, WE BEGIN WITH A CIRCLE OF RADIUS 1 ABOUT THE POINT $(x,y,z) = (2,0,0)$ IN THE XZ-PLANE.



NOW REVOLVE THIS ABOUT THE Z-AXIS TO OBTAIN A SURFACE IN $\mathbb{R}^3$.



EVERY POINT ON THE TORUS HAS COORDINATES (x, y, z) GIVEN BY

$$x = (2+\cos\theta)\cos\phi$$

$$y = (2+\cos\theta)\sin\phi$$

$$z = \sin\theta$$

FOR SOME $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$.

<u>EXERCISE</u> 71 : SHOW THAT THE CARTESIAN EQUATION OF THE TORUS CAN BE WRITTEN

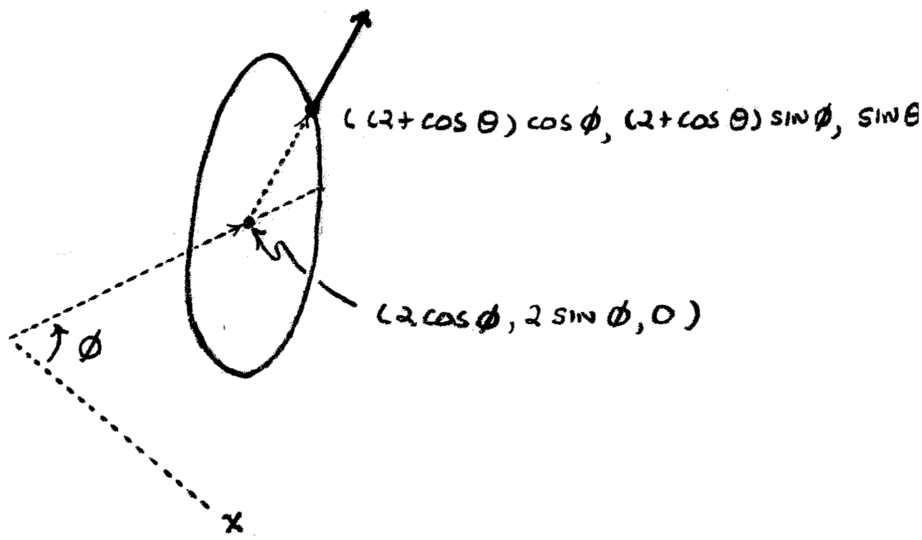
$$(x^2+y^2+z^2+3)^2 = 16(x^2+y^2)$$

NOW DEFINE $F : \mathbb{R}^3 \to \mathbb{R}$ BY

$$F(x,y,z) = (x^2+y^2+z^2+3)^2 - 16(x^2+y^2),$$

SHOW THAT $0 \in \mathbb{R}$ IS A REGULAR VALUE AND CONCLUDE THAT THE TORUS IS A SMOOTH SUBMANIFOLD OF $\mathbb{R}^3$.

GAUSS MAP :



UNIT NORMAL VECTOR HAS COMPONENTS

$$(\cos\theta\cos\phi\, ,\ \cos\theta\sin\phi\, ,\ \sin\theta)$$

THUS, THE GAUSS MAP G SHOULD ASSIGN TO THE POINT ON THE TORUS CORRESPONDING TO (θ, ϕ) THE POINT

$$(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta)$$

IN S^2.

TO PROVE SMOOTHNESS WE NEED CHARTS (U, φ) ON THE TORUS AND CHARTS (V, ψ) ON S^2 SO THAT WE CAN COMPUTE COORDINATE EXPRESSIONS $\psi \circ G \circ \varphi^{-1}$.

FOR THE CHARTS ON S^2 WE WILL USE SIMPLY THE PROJECTIONS INTO COORDINATE PLANES FROM VARIOUS OPEN HEMISPHERES.

ON THE TORUS WE WOULD LIKE TO USE (θ, ϕ) AS COORDINATES, BUT WE HAVE NOT YET SHOWN THAT THESE ARE, IN FACT, COORDINATE FUNCTIONS FOR ANY CHART ON THE TORUS. THIS IS ACTUALLY TRUE, BUT THE PROOF REQUIRES ONE ITEM (THE "INVERSE FUNCTION THEOREM") THAT WE WILL GET TO A BIT LATER. FOR THE TIME BEING WE WILL TAKE THE ARGUMENT ONLY UP TO THE POINT WHERE WE NEED A BIG GUN (AND WILL FILL IN THE LAST DETAIL WHEN WE HAVE IT).

CONSIDER THE MAP

$$\zeta : \mathbb{R}^2 \to \mathbb{R}^3$$

GIVEN BY

$$\xi(\theta,\phi) = \big((2+\cos\theta)\cos\phi,\ (2+\cos\theta)\sin\phi,\ \sin\theta\big)$$

THIS IS SMOOTH, OF COURSE, AND WE KNOW THAT IT MAPS ONTO THE TORUS. WE COMPUTE THE JACOBIAN OF THE MAP.

$$\begin{pmatrix} \frac{\partial \xi^1}{\partial\theta} & \frac{\partial \xi^1}{\partial\phi} \\ \frac{\partial \xi^2}{\partial\theta} & \frac{\partial \xi^2}{\partial\phi} \\ \frac{\partial \xi^3}{\partial\theta} & \frac{\partial \xi^3}{\partial\phi} \end{pmatrix} =$$

$$\begin{pmatrix} -\sin\theta\cos\phi & -(2+\cos\theta)\sin\phi \\ -\sin\theta\sin\phi & (2+\cos\theta)\cos\phi \\ \cos\theta & 0 \end{pmatrix}$$

THE POINT IS THAT THIS MATRIX HAS RANK 2 FOR ALL ϕ AND θ. BEING A MAP FROM $\mathbb{R}^2$ TO A 2-DIMENSIONAL MANIFOLD, THE "INVERSE FUNCTION THEOREM" WILL IMPLY THAT, ON A SUFFICIENTLY SMALL NEIGHBORHOOD OF ANY (θ,ϕ), ξ IS ONE-TO-ONE, MAPS ONTO AN OPEN NEIGHBORHOOD OF $\xi(\theta,\phi)$ ON THE TORUS AND HAS A SMOOTH INVERSE ξ^{-1} ON THIS NEIGHBORHOOD.

THUS, ON SOME OPEN NEIGHBORHOOD OF ANY POINT ON THE TORUS, $\varphi = \xi^{-1}$ IS A CHART WITH COORDINATE FUNCTIONS (θ, ϕ).

WITH THIS THE SMOOTHNESS OF THE GAUSS MAP G ON THE TORUS IS EASY SINCE WE CAN TAKE (U, φ) TO BE ANY OF THESE CHARTS ON THE TORUS AND (V, ψ) TO BE ANY PROJECTION INTO A COORDINATE PLANE FROM AN OPEN HEMISPHERE (SAY, INTO THE xy-PLANE FROM THE UPPER HEMISPHERE).

THEN

$$\begin{aligned}(\psi \circ G \circ \varphi^{-1})(\theta, \phi) &= (\psi \circ G \circ \xi)(\theta, \phi) \\ &= (\psi \circ G)((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta) \\ &= \psi(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) \\ &= (\cos\theta\cos\phi, \cos\theta\sin\phi)\end{aligned}$$

AND THE COORDINATE FUNCTIONS $\cos\theta\cos\phi$ AND $\cos\theta\sin\phi$ ARE SURELY SMOOTH FUNCTIONS OF θ AND ϕ.

3. (<u>THE GROUP OPERATIONS ON $GL(n, \mathbb{R})$</u>)

WE CONSIDER THE GENERAL LINEAR GROUP $GL(n, \mathbb{R})$ OF $n \times n$ INVERTIBLE REAL MATRICES.

THIS IS AN OPEN SUBMANIFOLD OF $\mathbb{R}^{n^2}$. THE PRODUCT MANIFOLD

$$GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \overset{\text{OPEN}}{\subseteq} \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \equiv \mathbb{R}^{2n^2}$$

IS THEN AN OPEN SUBMANIFOLD OF $\mathbb{R}^{2n^2}$. CONSIDER THE MAP

$$(A,B) \in GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow AB \in GL(n,\mathbb{R})$$

THE COORDINATE EXPRESSION FOR THIS MAP RELATIVE TO THE STANDARD CHARTS ON $GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \subseteq \mathbb{R}^{2n^2}$ AND $GL(n,\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ HAVE COORDINATE FUNCTIONS THAT ARE JUST POLYNOMIALS AND SO THE MAP IS SMOOTH.

SIMILARLY, THE MAP

$$A \in GL(n,\mathbb{R}) \longrightarrow A^{-1} \in GL(n,\mathbb{R})$$

IS SMOOTH.

THUS, $GL(n,\mathbb{R})$ IS AN EXAMPLE OF THE FOLLOWING : A <u>LIE</u> <u>GROUP</u> IS A GROUP G THAT ALSO HAS THE STRUCTURE OF A SMOOTH MANIFOLD FOR WHICH THE GROUP OPERATIONS

$$(a,b) \in G \times G \longrightarrow ab \in G$$

$$a \in G \longrightarrow a^{-1} \in G$$

ARE C^{∞}. WE WILL SEE MORE EXAMPLES AS WE PROCEED.

IF X AND Y ARE DIFFERENTIABLE MANIFOLDS AND $F: X \to Y$ IS A BIJECTION THAT IS SMOOTH WITH A SMOOTH INVERSE $F^{-1}: Y \to X$, THEN F IS A <u>DIFFEOMORPHISM</u> AND X AND Y ARE SAID TO BE <u>DIFFEOMORPHIC</u>.

THIS IS THE ANALOGUE FOR DIFFERENTIABLE MANIFOLDS OF "HOMEOMORPHISM" FOR TOPOLOGICAL SPACES, OR "ISOMORPHISM" FOR VECTOR SPACES, GROUPS, ETC.

IT IS NOT OBVIOUS, BUT WE WILL PROVE LATER THAT IF TWO MANIFOLDS ARE DIFFEOMORPHIC, THEN THEY MUST HAVE THE SAME DIMENSION.

BEFORE GETTING ON WITH THE GENERAL DEVELOPMENT I WANT TO PAUSE TO TELL YOU ABOUT ONE OF THE MOST AMAZING RESULTS TO COME OUT OF TOPOLOGY IN THE LAST 20 YEARS. FIRST YOU NEED TO SEE THAT A GIVEN TOPOLOGICAL SPACE CAN HAVE TWO <u>DIFFERENT</u> DIFFERENTIABLE STRUCTURES (MAXIMAL ATLASES) $\mathcal{A}$ AND $\mathcal{A}'$ FOR WHICH THE CORRESPONDING DIFFERENTIABLE MANIFOLDS X AND X' ARE <u>DIFFEOMORPHIC</u>.

<u>EXAMPLE</u>: LET X BE $\mathbb{R}$ WITH ITS STANDARD DIFFERENTIABLE STRUCTURE (ATLAS $\mathcal{A} = \{(\mathbb{R}, id)\}$) AND LET X' BE $\mathbb{R}$ WITH THE NONSTANDARD DIFFERENTIABLE STRUCTURE DETERMINED BY THE ATLAS $\mathcal{A}' = \{(\mathbb{R}, id^3)\}$. THESE ARE <u>DIFFERENT</u> DIFFERENTIABLE STRUCTURES. HOWEVER, id^3, <u>THOUGHT OF AS A MAP FROM</u> X' <u>TO</u> X, IS A DIFFEOMORPHISM. THIS FOLLOWS FROM

$$\begin{tikzcd} X' \arrow[r, "id^3"] & X \arrow[d, "id"] \\ \mathbb{R} \arrow[u, "\sqrt[3]{id}"] \arrow[r, dashed, "id"'] & \mathbb{R} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\sqrt[3]{id}"] & X' \arrow[d, "id^3"] \\ \mathbb{R} \arrow[u, "id"] \arrow[r, dashed, "id"'] & \mathbb{R} \end{tikzcd}$$

IN FACT, ONE CAN PROVE THAT ANY TWO DIFFERENTIABLE STRUCTURES ON THE TOPOLOGICAL SPACE $\mathbb{R}$ ARE NECESSARILY DIFFEOMORPHIC.

A MUCH MORE DIFFICULT RESULT IS THE

THEOREM: ANY TWO DIFFERENTIABLE STRUCTURES ON THE TOPOLOGICAL SPACES $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \quad \mathbb{R}^5, \mathbb{R}^6, \ldots$ ARE NECESSARILY DIFFEOMORPHIC.

INFINITELY MORE DIFFICULT STILL IS THE FOLLOWING RESULT, PROVED AROUND 1983.

THEOREM: THERE ARE $2^{\aleph_0}$ NONDIFFEOMORPHIC DIFFERENTIABLE STRUCTURES ON THE TOPOLOGICAL SPACE $\mathbb{R}^4$.

THIS THEOREM RESULTED FROM THE COMBINATION OF TWO VERY DIFFERENT, BUT EQUALLY PROFOUND AND REVOLUTIONARY DEVELOPMENTS IN TOPOLOGY, BOTH OF WHICH OCCURED IN 1982 (ONE DUE TO MICHAEL FREEDMAN AND THE OTHER DUE TO SIMON DONALDSON).

ALL I WILL SAY ABOUT THIS IS THAT DONALDSON'S WORK RESULTED FROM A DEEP ANALYTICAL STUDY OF THE SPACE OF SOLUTIONS TO A CERTAIN PDE THAT AROSE IN PHYSICS (YANG-MILLS EQUATIONS)

EXERCISE 72 : LET G_1 AND G_2 BE LIE GROUPS. PROVIDE $G_1 \times G_2$ WITH THE PRODUCT TOPOLOGY, THE PRODUCT MANIFOLD STRUCTURE AND THE DIRECT PRODUCT GROUP STRUCTURE. SHOW THAT $G_1 \times G_2$ IS A LIE GROUP.

EXERCISE 73 : LET G BE LIE GROUP AND LET g BE A FIXED ELEMENT OF G. DEFINE MAPS

$$L_g : G \to G \qquad L_g(h) = gh$$

$$R_g : G \to G \qquad R_g(h) = hg$$

CALLED LEFT (L_g) AND RIGHT (R_g) TRANSLATION BY g. SHOW THAT THESE ARE DIFFEOMORPHISMS OF G ONTO ITSELF.

EXERCISE 74 : LET G BE A LIE GROUP AND LET e DENOTE ITS IDENTITY ELEMENT. LET G_e DENOTE THE CONNECTED COMPONENT OF G CONTAINING e. SHOW THAT G_e IS A NORMAL SUBGROUP OF G.

HINTS : TO SHOW THAT G_e IS CLOSED UNDER THE FORMATION OF INVERSES, FIX $h \in G_e$ AND SHOW THAT $L_{h^{-1}}(G_e) = G_e$. FOR PRODUCTS, LET $h, k \in G_e$ AND SHOW THAT $L_h(G_e) = G_e$. TO SHOW THAT G_e IS NORMAL (I.E., $g\, G_e\, g^{-1} = G_e \;\forall g \in G$) CONSIDER $R_{g^{-1}} \circ L_g (G_e)$.