# "TOPOLOGICAL" RESULTS IN GENERAL RELATIVITY

# THE SINGULARITY THEOREMS

**Greg Naber**

**"THEOREM": ANY CLASSICAL (NON-QUANTUM) MODEL OF THE UNIVERSE WHICH**

1. **CONTAINS NO CAUSALITY VIOLATIONS,**
2. **IS DETERMINISTIC,**
3. **MODELS AN EXPANDING UNIVERSE, AND**
4. **HAS THE PROPERTY THAT GRAVITY IS ALWAYS ATTRACTIVE**

**MUST BE SINGULAR, I.E., CONTAIN REGIONS IN WHICH THE LAWS OF PHYSICS AS WE CURRENTLY KNOW THEM MUST BREAK DOWN.**

**THEOREM: LET M BE A SPACETIME WHICH SATISFIES EACH OF THE FOLLOWING CONDITIONS:**

1. **M IS STABLY CAUSAL,**
2. **M IS GLOBALLY HYPERBOLIC,**
3. **M CONTAINS A CAUCHY SURFACE ON WHICH THE MEAN CURVATURE IS BOUNDED BELOW BY SOME POSITIVE CONSTANT, AND**
4. **RIC (V,V) $\geq$ 0 FOR ALL TIMELIKE VECTORS V.**

**THEN M IS TIMELIKE GEODESICALLY INCOMPLETE.**

**HAWKING'S THEOREM:** Let M be a spacetime which satisfies each of the following conditions:

1. M is stably causal,
2. M is globally hyperbolic,
3. M contains a Cauchy surface on which the mean curvature is bounded below by some positive constant, and
4. Ric (V,V) ≥ 0 for all timelike vectors V.

Then M is timelike geodesically incomplete.

FOR THOSE WHO KNOW THE JARGON :

A SPACETIME IS A 4-DIMENSIONAL, CONNECTED, SMOOTH MANIFOLD WITH A LORENTZ METRIC.

FOR THOSE WHO DO NOT :

I WILL DESCRIBE THE SIMPLEST EXAMPLE ( " MINKOWSKI SPACETIME " ), HOW ALL OTHER EXAMPLES DIFFER FROM IT, AND THEN A FEW CONCRETE EXAMPLES OF THESE MORE GENERAL OBJECTS.

1905 " ON THE ELECTRODYNAMICS OF MOVING BODIES "

ALBERT EINSTEIN

1907 " SPACE AND TIME "

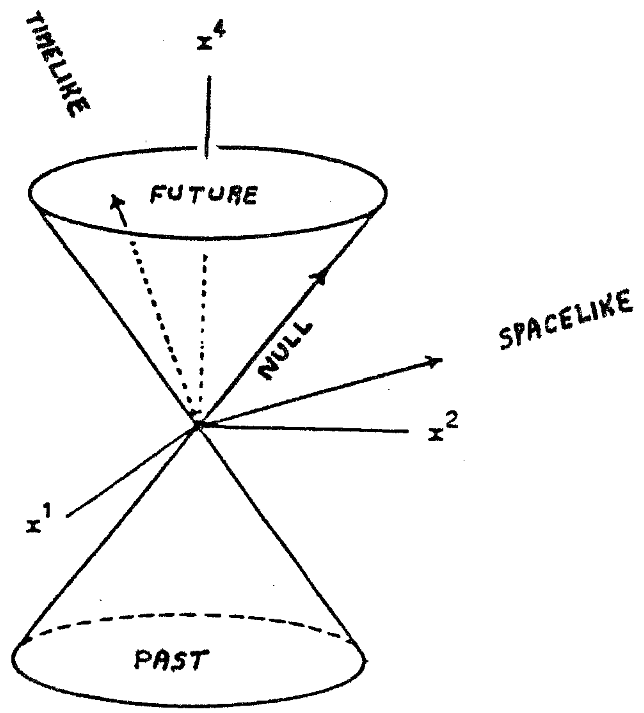
HERMANN MINKOWSKI

MINKOWSKI SPACETIME $\mathcal{M}$ :

VECTOR SPACE $\mathbb{R}^4$ : POINTS $x = (x^1, x^2, x^3, x^4)$ ARE "EVENTS" WITH SPATIAL $(x^1, x^2, x^3)$ AND TIME $(x^4)$ COORDINATES SUPPLIED BY SOME INERTIAL OBSERVER, BUT WITH TIME MEASURED IN UNITS OF DISTANCE $(x^4 = ct)$.

"INNER PRODUCT" : $\langle x, y \rangle = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4$

NULL CONE : $(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = 0$




NULL CONE AT $p = (p^1, p^2, p^3, p^4)$ : $(x^1-p^1)^2 + (x^2-p^2)^2 + (x^3-p^3)^2 - (x^4-p^4)^2 = 0$

SMOOTH CURVES IN $\mathcal{M}$ ARE <u>NULL</u>, <u>TIMELIKE</u>, OR <u>SPACELIKE</u> IF THEIR TANGENT VECTORS LIE ON, INSIDE, OR OUTSIDE THE NULL CONE AT EACH POINT.

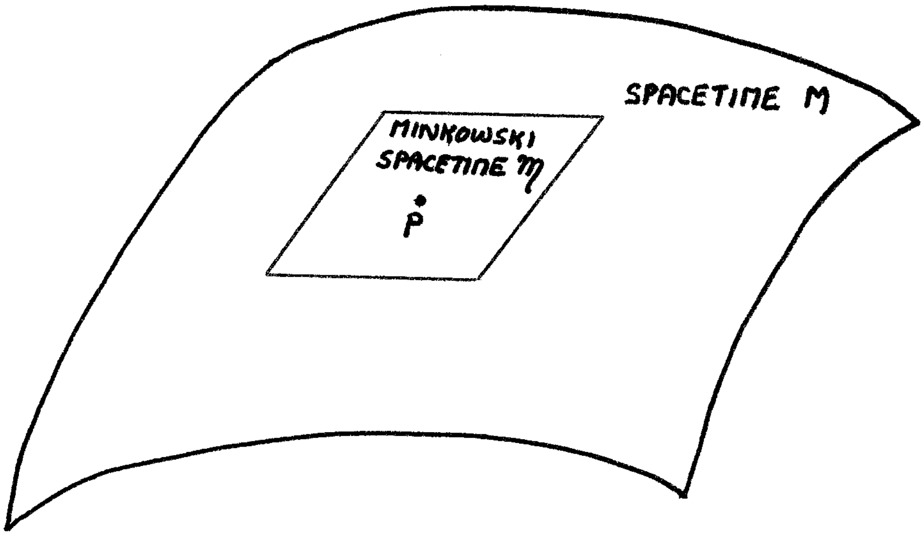
FUTURE-DIRECTED, TIMELIKE CURVES ARE CALLED <u>WORLDLINES</u>.

FUTURE-DIRECTED, TIMELIKE STRAIGHT LINES ARE CALLED <u>FREE</u> <u>WORLDLINES</u>.

MINKOWSKI SPACETIME IS A MODEL OF THE EVENT WORLD WHEN GRAVITATIONAL EFFECTS ARE CONSIDERED NEGLIGIBLE.

EINSTEIN'S "ELEVATOR EXPERIMENT"

$\Rightarrow$ IN THE PRESENCE OF GRAVITATIONAL FIELDS, THE EVENT WORLD (SPACETIME) IS "LOCALLY LIKE MINKOWSKI SPACETIME" IN THE SAME SENSE THAT A SMOOTH SURFACE IN SPACE IS LOCALLY LIKE THE PLANE.




AT EACH $p$ IN $M$ THERE IS A TANGENT SPACE WITH ALL THE STRUCTURE OF $\mathcal{M}$

$M$ IS A "4-DIMENSIONAL SMOOTH MANIFOLD" WITH MINKOWSKI INNER PRODUCTS ON THE TANGENT SPACES THAT VARY SMOOTHLY FROM POINT TO POINT ("LORENTZ METRIC")

TWO SIMPLE EXAMPLES :

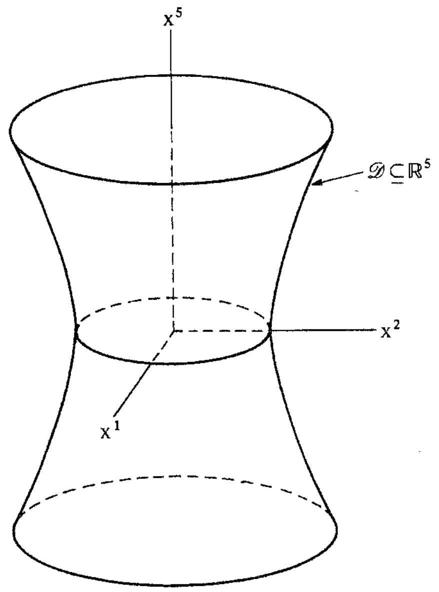
1. ( DE SITTER SPACETIME $\mathcal{D}$ )

BEGIN WITH $\mathbb{R}^5$ AND ITS "MINKOWSKI-LIKE" INNER PRODUCT

$$\langle x, y \rangle = x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4 - x^5y^5$$

AND CONSIDER THE SUBSET $\mathcal{D}$ ANALOGOUS TO THE HYPERBOLOID OF ONE SHEET $x^2+y^2-z^2 = 1$ IN $\mathbb{R}^3$ :

$$\mathcal{D} : \quad (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 - (x^5)^2 = 1$$



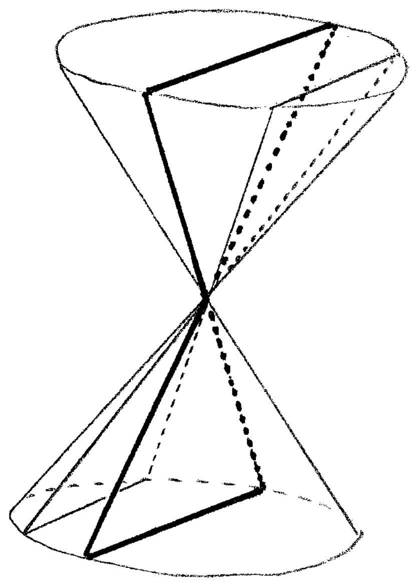

THE "CIRCLES" HERE ARE REALLY 3-SPHERES $S^3$.

AS A MANIFOLD, $\mathcal{D} = S^3 \times \mathbb{R}$ .

THE TANGENT SPACE TO $\mathcal{D}$ AT SOME POINT $p$ CONSISTS OF ALL VELOCITY VECTORS AT $p$ TO SMOOTH CURVES IN $\mathcal{D}$ THROUGH $p$.

RESTRICTING THE "MINKOWSKI-LIKE" INNER PRODUCT OF $\mathbb{R}^5$ TO EACH SUCH TANGENT SPACE GIVES A LORENTZ METRIC ON $\mathcal{D}$.

THE NULL CONE AT ANY $p \in \mathcal{D}$ IS THEREFORE THE INTERSECTION OF THE "MINKOWSKI-LIKE" NULL CONE AT $p$ IN $\mathbb{R}^5$ WITH THE TANGENT SPACE TO $\mathcal{D}$ AT $p$.



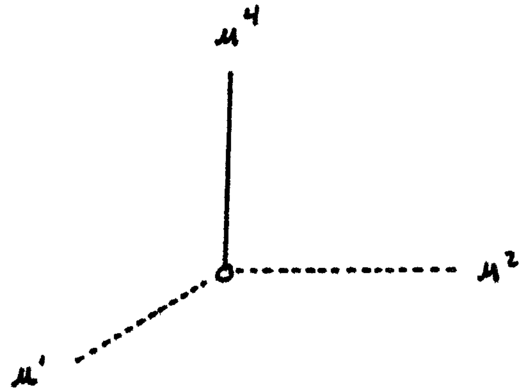
CLASSICALLY, $\mathcal{D}$ WAS REGARDED AS SOMETHING OF A "TOY MODEL", BUT THIS MAY BE CHANGING.

$\mathcal{D}$ DOES (APPARENTLY) PLAY A SIGNIFICANT ROLE IN QUANTUM GRAVITY AND CONFORMAL FIELD THEORY.

2. (<u>EINSTEIN</u> - <u>DE SITTER</u> <u>SPACETIME</u> $\mathcal{E}$)

THIS IS AN HONEST COSMOLOGICAL MODEL (SOLUTION TO THE "EINSTEIN FIELD EQUATIONS" FOR A UNIFORM "DUST" OF GALAXIES).

$$\mathcal{E} = \mathbb{R}^3 \times (0,\infty)$$




THE TANGENT SPACE AT ANY $p = (u^1, u^2, u^3, u^4)$ IN $\mathcal{E}$ IS THE SAME AS IT IS IN $\mathbb{R}^4$.

HOWEVER, THE INNER PRODUCT WE INTRODUCE ON THIS TANGENT SPACE DEPENDS ON $p$.

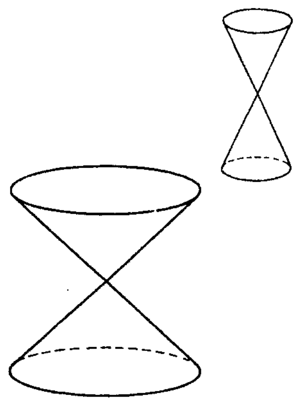
IF $v_p$ AND $w_p$ ARE TANGENT VECTORS AT $p$
WE DEFINE

$$\langle v_p, w_p \rangle = (u^4)^{\frac{4}{3}} (v^1 w^1 + v^2 w^2 + v^3 w^3) - v^4 w^4$$

NULL CONES

$$(u^4)^{\frac{4}{3}} \left( (v^1)^2 + (v^2)^2 + (v^3)^2 \right) - (v^4)^2 = 0$$

GET "STEEPER" AS $p$ "GETS HIGHER":



**HAWKING'S THEOREM: Let M be a spacetime which satisfies each of the following conditions:**

1. **M is stably causal,**
2. **M is globally hyperbolic,**
3. **M contains a Cauchy surface on which the mean curvature is bounded below by some positive constant, and**
4. **Ric (V,V) ≥ 0 for all timelike vectors V.**

**Then M is timelike geodesically incomplete.**

CAUSALITY VIOLATIONS IN M : E.G.,

1. CLOSED TIMELIKE CURVES
2. CLOSED NULL CURVES
3. TIMELIKE CURVES WHICH, ALTHOUGH NOT CLOSED, CONTINUALLY ENTER, LEAVE AND RE-ENTER ARBITRARILY SMALL NEIGHBORHOODS OF A POINT

NO GOOD CAN COME OF THIS.

HAWKING FOUND A PHYSICALLY REASONABLE ASSUMPTION WHICH PROHIBITS ALL SUCH REPREHENSIBLE BEHAVIOR

IDEA : THE LORENTZ METRIC IS FUNDAMENTALLY A MEASURED QUANTITY. ANY ASSUMPTION MADE ABOUT IT SHOULD BE "STABLE" (I.E., INSENSITIVE TO SMALL PERTURBATIONS).

A SPACETIME $M$ WITH LORENTZ METRIC $g$ IS <u>STABLY CAUSAL</u> IF THERE ARE NO CLOSED TIMELIKE CURVES IN ANY LORENTZ METRIC FOR $M$ THAT IS "CLOSE" TO $g$ (WITH "CLOSE" DEFINED IN TERMS OF SOME APPROPRIATE TOPOLOGY ON THE SET OF ALL LORENTZ METRICS FOR $M$).

MORE CONVENIENT FORMULATION :

> <u>RECALL</u> : IN CALCULUS, THE GRADIENT $\nabla f$ OF A REAL-VALUED FUNCTION $f$ ON $\mathbb{R}^3$ IS CHARACTERIZED BY THE PROPERTY THAT "DOTTING" IT WITH A VECTOR GIVES RATE OF CHANGE OF $f$ WITH RESPECT TO THAT VECTOR (IN THE DIRECTION OF THE VECTOR) AND HAS THE PROPERTY THAT IT IS ORTHOGONAL TO THE LEVEL SURFACES $f$ = CONSTANT AT EACH POINT.

WHENEVER THERE IS A METRIC (E.G., LORENTZ METRIC) PRESENT IT IS POSSIBLE TO DEFINE AN ANALOGOUS GRADIENT WITH THESE SAME PROPERTIES.

<u>THEOREM</u> (HAWKING): A SPACETIME $M$ IS STABLY CAUSAL IF AND ONLY IF THERE IS DEFINED ON $M$ A SMOOTH REAL-VALUED FUNCTION $T$ WHOSE GRADIENT $\nabla T$ IS EVERYWHERE TIMELIKE ($\langle \nabla T, \nabla T \rangle < 0$).

SUCH A FUNCTION $T$ IS CALLED A GLOBAL TIME FUNCTION ON $M$ (DESPITE THE FACT THAT ITS VALUES GENERALLY HAVE NO PHYSICAL SIGNIFICANCE AT ALL).

EXAMPLES: THE $4^{TH}$ COORDINATE FUNCTIONS $x^4$ ON $\mathcal{M}$ AND $u^4$ ON $\mathcal{E}$ ARE GLOBAL TIME FUNCTIONS. ON $\mathcal{D}$ ONE CAN PRODUCE A GLOBAL TIME FUNCTION AS FOLLOWS: THE EQUATIONS

$$x^1 = \cosh(y^4)\cos(y^1)$$
$$x^2 = \cosh(y^4)\sin(y^1)\cos(y^2)$$
$$x^3 = \cosh(y^4)\sin(y^1)\sin(y^2)\cos(y^3)$$
$$x^4 = \cosh(y^4)\sin(y^1)\sin(y^2)\sin(y^3)$$
$$x^5 = \sinh(y^4)$$

DEFINE COORDINATES $(y^1, y^2, y^3, y^4)$ ON $\mathcal{D}$ AND $y^4$ IS A GLOBAL TIME FUNCTION.

THE LEVEL HYPERSURFACES

$$T = \text{CONSTANT}$$

OF $T$ IN $M$ ARE CALLED SPACELIKE SLICES IN $M$.

ABSTRACT MODELS OF "INSTANTANEOUS 3-SPACE".

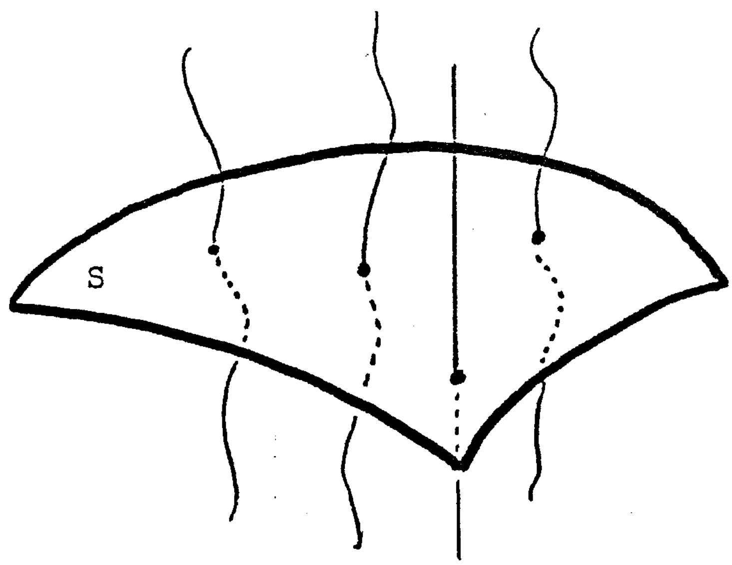
**HAWKING'S THEOREM:** Let M be a spacetime which satisfies each of the following conditions:

1. M is stably causal,
2. M is globally hyperbolic,
3. M contains a Cauchy surface on which the mean curvature is bounded below by some positive constant, and
4. Ric (V,V) $\geq$ 0 for all timelike vectors V.

Then M is timelike geodesically incomplete.

THE EXISTENCE OF A TIMELIKE CURVE CONNECTING TWO EVENTS ESTABLISHES A "CAUSAL CONNECTION" BETWEEN THE EVENTS.

A SPACELIKE SLICE $S$ IN $M$ IS A CAUCHY SURFACE FOR $M$ IF EVERY MAXIMAL TIMELIKE CURVE IN $M$ INTERSECTS $S$ PRECISELY ONCE.



THESE EXIST IN SOME SPACETIMES (E.G., $\mathcal{M}$, $\mathcal{D}$ AND $\mathcal{E}$), BUT NOT IN OTHERS (E.G., $\mathcal{M} - \{(0,0,0,0)\}$)

M IS SAID TO BE <u>GLOBALLY HYPERBOLIC</u> IF IT HAS A CAUCHY SURFACE.

GLOBAL HYPERBOLICITY IS REGARDED AS A VERY STRONG FORM OF THE PRINCIPLE OF DETERMINISM: IF S IS GIVEN BY $T = t_0$, THEN

> "EVERYTHING THAT HAPPENS 'BEFORE' $T = t_0$
> IS REGISTERED ON S AND DETERMINES
> EVERYTHING THAT HAPPENS 'AFTER' $T = t_0$."

**HAWKING'S THEOREM: Let M be a spacetime which satisfies each of the following conditions:**

1. **M is stably causal,**
2. **M is globally hyperbolic,**
3. **M contains a Cauchy surface on which the <u>mean</u> <u>curvature</u> is bounded below by some positive constant, and**
4. **Ric (V,V) $\geq 0$ for all timelike vectors V.**

**Then M is timelike geodesically incomplete.**

RECALL: IN CALCULUS, THE DIVERGENCE OF A VECTOR FIELD REPRESENTS THE LOCAL RATE OF EXPANSION OR CONTRACTION PER UNIT VOLUME (E.G., FOR FLUID FLOW).

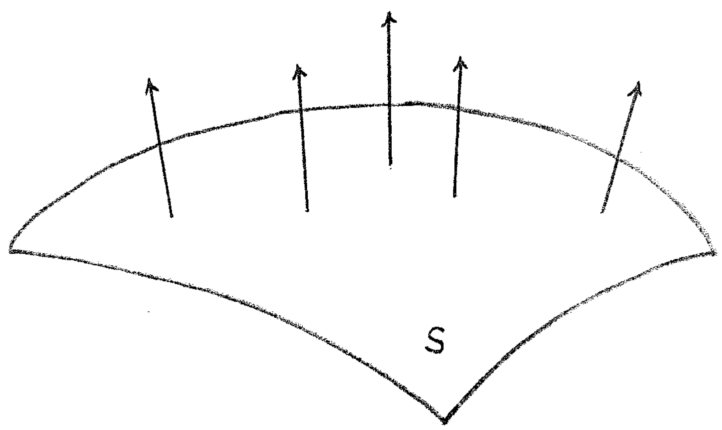
IT IS POSSIBLE TO DEFINE AN ANALOGOUS NOTION WHENEVER THERE IS A METRIC (E.G., LORENTZ METRIC) PRESENT.

LET $S$ $(T = t_0)$ BE A CAUCHY SURFACE IN $M$.

$\nabla T$ IS NORMAL TO $S$

$$N = -\frac{1}{\sqrt{|\langle \nabla T, \nabla T\rangle|}}\nabla T$$

= UNIT NORMAL TO $S$




$$H_S = \operatorname{div}(N|S)$$

= MEAN CURVATURE OF $S$

EXAMPLES :

1. $\mathcal{M}$ $\qquad S(x^4 = x_0^4)$ $\qquad H_S(p) = 0 \quad \forall p \in S$

2. $\mathcal{D}$ $\qquad S(y^4 = y_0^4)$ $\qquad H_S(p) = \dfrac{1}{\cosh(y_0^4)} \quad \forall p \in S$

3. $\mathcal{E}$ $\qquad S(\mu^4 = \mu_0^4)$ $\qquad H_S(p) = \dfrac{2}{\mu_0^4} \quad \forall p \in S$

THE THIRD HYPOTHESIS OF HAWKING'S THEOREM IS

> THERE EXISTS A CAUCHY SURFACE $S$
> SUCH THAT, FOR SOME POSITIVE
> CONSTANT $K$,
>
> $$H_S(p) \geq K$$
>
> $\forall p \in S$.

AT SOME "INSTANT" THE UNIVERSE
IS STRICTLY EXPANDING.

THIS IS SATISFIED BY $\mathcal{D}$ AND $\mathcal{E}$, BUT NOT $\mathcal{M}$.

**HAWKING'S THEOREM: Let M be a spacetime which satisfies each of the following conditions:**

1. **M is stably causal,**
2. **M is globally hyperbolic,**
3. **M contains a Cauchy surface on which the mean curvature is bounded below by some positive constant, and**
4. **Ric (V,V) $\geq$ 0 for all timelike vectors V.**

**Then M is timelike geodesically incomplete.**

EINSTEIN FIELD EQUATIONS

$$Ric - \frac{1}{2} S g = 8\pi T$$

PLUS

STRONG ENERGY CONDITION ON $T$

IMPLY

$$Ric(V,V) \geq 0 \quad \forall \text{ TIMELIKE } V$$

THUS, THE FOURTH HYPOTHESIS OF HAWKING'S THEOREM IS ESSENTIALLY A GEOMETRICAL VERSION (VIA THE EINSTEIN EQUATIONS) OF A PHYSICALLY REASONABLE ASSUMPTION ON ENERGY - MOMENTUM TENSOR WHICH GIVES RISE TO THE FIELD.

NOTE : THE EINSTEIN EQUATIONS THEMSELVES ARE NOT ASSUMED IN HAWKING'S THEOREM - ONLY THIS ONE RATHER WEAK CONSEQUENCE OF THEM.

PROOF (FOR THOSE FAMILIAR WITH THE EINSTEIN EQUATIONS):

IN LOCAL COORDINATES THE EINSTEIN EQUATIONS READ

$$R_{ab} - \tfrac{1}{2} S g_{ab} = 8\pi T_{ab} \tag{1}$$

WHERE $S = g^{ab} R_{ab}$ IS THE SCALAR CURVATURE. THE STRONG ENERGY CONDITION ON $T_{ab}$ REQUIRES THAT, FOR TIMELIKE $V^a$,

$$T_{ab} V^a V^b \geq \tfrac{1}{2} (\operatorname{tr} T) g_{ab} V^a V^b \tag{2}$$

WHERE $\operatorname{tr} T = g^{ab} T_{ab}$. MULTIPLYING (1) ON BOTH SIDES BY $g^{ab}$ AND SUMMING AS INDICATED GIVES

$$S - \tfrac{1}{2} S \underbrace{(g^{ab} g_{ab})}_{4} = 8\pi \operatorname{tr} T$$

$$-S = 8\pi \operatorname{tr} T$$

$$S = -8\pi \operatorname{tr} T$$

THUS,

$$R_{ab} + 4\pi \operatorname{tr} T\, g_{ab} = 8\pi T_{ab}$$

AND, FOR TIMELIKE $V^a$,

$$\begin{aligned} \operatorname{Ric}(V,V) = R_{ab} V^a V^b &= 8\pi T_{ab} V^a V^b - 4\pi \operatorname{tr} T\, g_{ab} V^a V^b \\ &= 8\pi \left( T_{ab} V^a V^b - \tfrac{1}{2} \operatorname{tr} T\, g_{ab} V^a V^b \right) \\ &\geq 0. \end{aligned}$$

□

EXAMPLES :

1. $\mathcal{M}$ RIC IS IDENTICALLY ZERO SO THE CONDITION IS TRIVIALLY SATISFIED.

2. $\mathcal{D}$ $\mathrm{Ric} = 3\langle\ ,\ \rangle$ SO, FOR TIMELIKE $V$, $\mathrm{Ric}(V,V) = 3\langle V,V\rangle < 0$ AND THE CONDITION IS NOT SATISFIED.

3. $\mathcal{E}$ AT ANY POINT $P = (P^1, P^2, P^3, P^4)$ AND FOR ANY TIMELIKE VECTOR $V$ AT $P$,

$$\mathrm{Ric}(V,V) = \frac{4}{3}\left(\frac{V^4}{P^4}\right)^2 > 0$$

AND SO THE CONDITION IS SATISFIED.

NOTE : FOR FUTURE REFERENCE, THE SCALAR CURVATURE OF $\mathcal{E}$ IS

$$S = \frac{4}{3}\left(\frac{1}{P^4}\right)^2 .$$

WITH THIS RUDIMENTARY UNDER STANDING OF WHAT THE HYPOTHESES OF HAWKING'S THEOREM SAY, WE MOVE ON TO THE SENSE IN WHICH THEY IMPLY THAT THE SPACETIME IS " SINGULAR ".
KEEP IN MIND THAT BOTH $\mathcal{M}$ AND $\mathcal{D}$ VIOLATE ONE OF THE HYPOTHESES, BUT $\mathcal{E}$ SATISFIES ALL OF THEM.

**HAWKING'S THEOREM:** Let M be a spacetime which satisfies each of the following conditions:

1. M is stably causal,
2. M is globally hyperbolic,
3. M contains a Cauchy surface on which the mean curvature is bounded below by some positive constant, and
4. Ric (V,V) ≥ 0 for all timelike vectors V.

Then M is timelike geodesically incomplete.

USUALLY IN PHYSICS SOMETHING IS "SINGULAR" IF IT BLOWS UP SOMEWHERE, E.G.,

(1) THE COULOMB FIELD AT THE LOCATION OF THE CHARGE.

(2) (FOR THOSE FAMILIAR WITH GR) THE SCHWARTZSCHILD METRIC AT "$r = 2m$".

(3) THE SCALAR CURVATURE $S = \frac{4}{3}\left(\frac{1}{p^4}\right)^2$ IN $\mathcal{E}$ AS $p^4 \to 0$.

SOMETIMES THESE ARE "REAL" PHYSICAL SINGULARITIES (E.G., (3)), SOMETIMES THEY REFLECT A DEFICIENCY IN THE MATHEMATICAL MODEL (E.G., (1)), AND SOMETIMES THEY ARE DUE SIMPLY TO AN UNFORTUNATE CHOICE OF COORDINATES (E.G., (2)).

WHEN THE "THING" THAT BECOMES SINGULAR IS THE METRIC OF SPACETIME ITSELF (E.G., (2)) IT CAN BE VERY DIFFICULT TO DECIDE BETWEEN THESE POSIBILITIES

E.G., THE SPACETIME $M = \mathbb{R}^3 \times (0,\infty)$ WITH METRIC AT EACH $p = (p^1, p^2, p^3, p^4)$ GIVEN BY

$$\langle v_p, w_p \rangle_p = v^1 w^1 + v^2 w^2 + v^3 w^3 - \frac{1}{(p^4)^2} v^4 w^4$$

LOOKS "BAD" AS $p^4 \to 0$, BUT IS REALLY JUST A DISGUISED VERSION OF $\mathcal{M}$.

BESIDES, THE VERY DEFINITIONS OF "MANIFOLD" AND "LORENTZ METRIC" REQUIRE SMOOTHNESS - "BAD" POINTS MUST ALREADY HAVE BEEN CUT OUT.

THE TRICK THEN IS TO DETECT THE "HOLES" THAT REMAIN AFTER THE BAD POINTS HAVE BEEN REMOVED. HOW DO YOU DETECT A "HOLE" IN SPACETIME ?

SIMPLE ! THERE'S A "HOLE" THERE IF SOMETHING FALLS THROUGH IT !

A SPACETIME IS <u>MAXIMAL</u> IF IT IS NOT PROPERLY CONTAINED IN ANY OTHER SPACETIME (E.G., $\mathcal{M}$, $\mathcal{D}$ AND $\mathcal{E}$, BUT NOT $\mathcal{M} - \{(0,0,0,0)\}$).

WE WILL REGARD A SPACETIME AS <u>SINGULAR</u> IF IT IS MAXIMAL AND TIMELIKE GEODESICALLY INCOMPLETE.

**HAWKING'S THEOREM: Let M be a spacetime which satisfies each of the following conditions:**

1. **M is stably causal,**
2. **M is globally hyperbolic,**
3. **M contains a Cauchy surface on which the mean curvature is bounded below by some positive constant, and**
4. **Ric (V,V) $\geq$ 0 for all timelike vectors V.**

**Then M is timelike geodesically incomplete.**

AND SO, IF IT IS MAXIMAL, IT IS SINGULAR

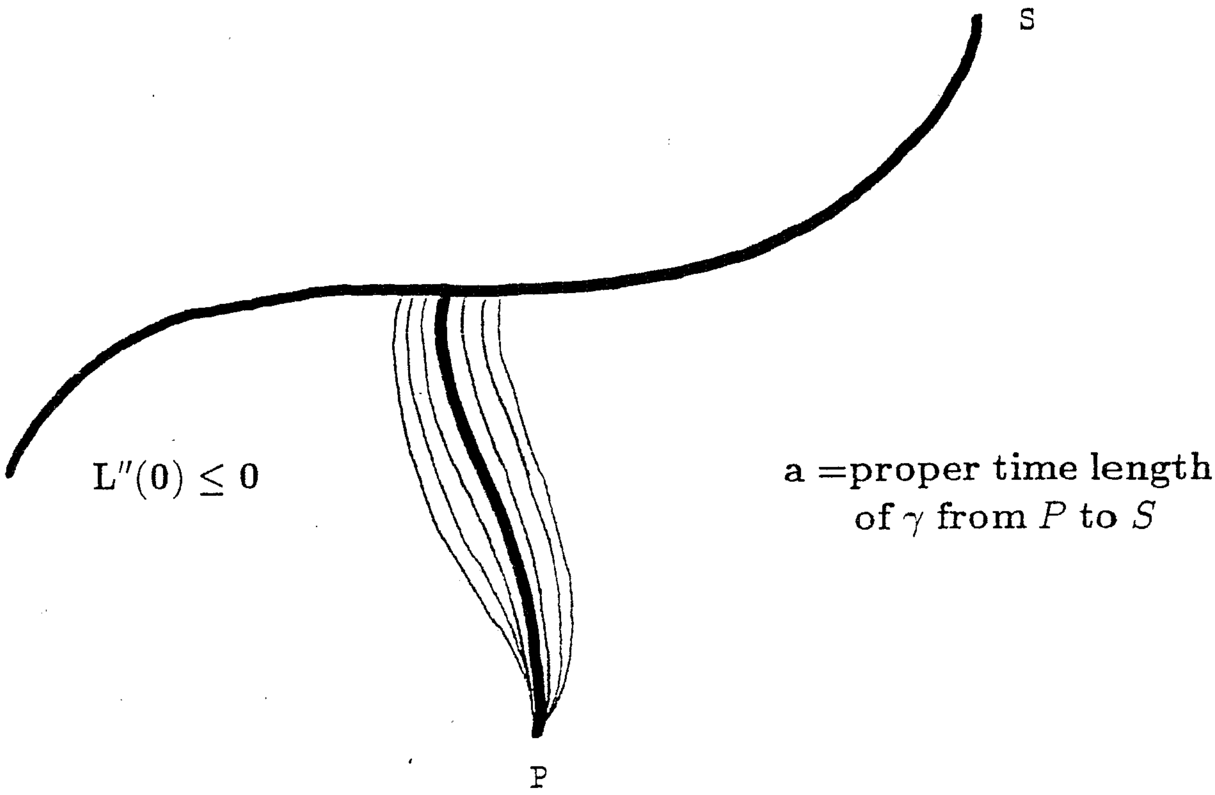
HERE'S A PICTURE OF THE PROOF :

"SOMETHING FALLS THROUGH IT" MEANS

> SOME FREE MATERIAL PARTICLE HAS A WORLDLINE WHICH SIMPLY CEASES TO EXIST AFTER A FINITE LAPSE OF ITS OWN (PROPER) TIME.

TO MAKE THIS PRECISE WE NEED A BIT OF TERMINOLOGY.

> THE WORLDLINE OF A MATERIAL PARTICLE IS A TIMELIKE CURVE, WHICH CAN BE PARAMETRIZED BY THE TIME READ FROM A CLOCK CARRIED ALONG BY THE PARTICLE (THE PARTICLE'S *PROPER TIME*).
>
> THE WORLDLINE OF A FREE MATERIAL PARTICLE IS A *TIMELIKE GEODESIC*.
>
> A TIMELIKE GEODESIC IS *COMPLETE* IF ITS PROPER TIME PARAMETRIZATION IS DEFINED ON $(-\infty, \infty)$ AND *INCOMPLETE* OTHERWISE.
>
> A SPACETIME IS *TIMELIKE GEODESICALLY COMPLETE* IF EACH OF ITS MAXIMAL TIMELIKE GEODESICS IS COMPLETE AND *TIMELIKE GEODESICALLY INCOMPLETE* OTHERWISE.




$$0 \leq \frac{3}{a} - \int_{-a}^{0} \left(\frac{a+u}{a}\right)^2 \mathrm{Ric}(\gamma', \gamma')\,du - H_S(\gamma(0))$$

$$0 \leq \frac{3}{a} - H_S(\gamma(0))$$

$$0 \leq \frac{3}{a} - K$$

$$a \leq \frac{3}{K}$$

AND HERE'S A BRIEF EXPLANATION :

$M$ = THE SPACETIME

$S$ = THE CAUCHY SURFACE ON WHICH THE MEAN CURVATURE $H_S$ IS BOUNDED BELOW BY SOME POSITIVE CONSTANT $K$

GLOBAL HYPERBOLICITY $\Rightarrow$ FOR EVERY $p \in M - S$ THERE IS A SMOOTH TIMELIKE CURVE $\gamma$ FROM $p$ TO $S$ THAT MAXIMIZES THE PROPER TIME LENGTH AMONG SMOOTH TIMELIKE CURVES JOINING $p$ TO $S$.

MUST BE A GEODESIC AND INTERSECT $S$ ORTHOGONALLY.

FOR ANY 1-PARAMETER FAMILY OF SMOOTH TIMELIKE CURVES $\{\gamma^s(t)\}$ JOINING $p$ TO $S$ WITH $\gamma^0 = \gamma$, LET

$$L(s) = \text{PROPER TIME LENGTH OF } \gamma^s \text{ FROM } p \text{ TO } S$$

THEN

$$L''(0) \leq 0 .$$

"SECOND VARIATION FORMULA" FOR $L''(0)$ YIELDS FIRST INEQUALITY ON PREVIOUS PAGE.

$$\vdots$$

$$\alpha \leq \frac{3}{K}$$

$\gamma$ CANNOT BE EXTENDED INTO THE PAST BEYOND $\frac{3}{K}$ SO $\gamma$ IS INCOMPLETE.