MATH 680 LECTURE 15 THE TANGENT BUNDLE

CHAPTER 4, GUDMUNDSSON

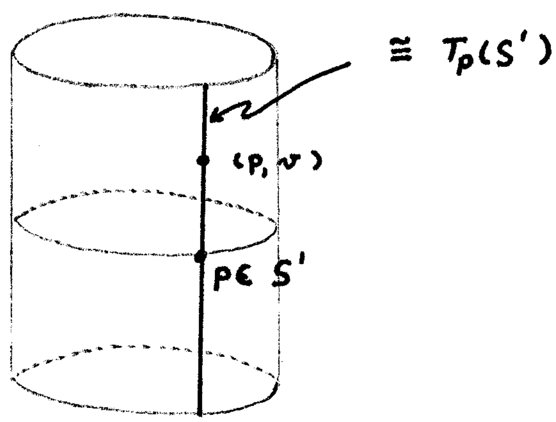
RECALL: IN AN EARLIER EXERCISE (LECTURE 10, P. 15) YOU SHOWED THAT

$$\{ (p, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|p\| = 1, \; p \cdot v = 0 \}$$

IS A SUBMANIFOLD OF $\mathbb{R}^{2n+2}$ OF DIMENSION $2n$ AND WE VISUALIZED THIS AS THE COLLECTION OF ALL PAIRS

$$( p \in S^n, \; v \in T_p(S^n) )$$

GLUED TOGETHER INTO A MANIFOLD, E.G., FOR $n = 1$,




IT LOOKS LIKE (AND, IN FACT, IS DIFFEOMORPHIC TO) THE CYLINDER $S^1 \times \mathbb{R}$.

DENOTE THIS MANIFOLD $TS^n$. NOTE THAT THERE IS A NATURAL PROJECTION MAP

$$\pi : TS^n \longrightarrow S^n$$

$$\pi(p, v) = p$$

THAT IS CLEARLY SMOOTH

MOREOVER, FOR EVERY $p \in S^n$,

$$\pi^{-1}(p)$$

IS A SUBSET OF $TS^n$ (THE "FIBER ABOVE $p$") THAT HAS A NATURAL VECTOR SPACE STRUCTURE (I.E., IT IS A COPY OF THE TANGENT SPACE TO $S^n$ AT $p$).

NOTICE THAT A SMOOTH MAP

$$s : S^n \to TS^n$$

THAT SATISFIES

$$\pi \circ s = id_{S^n}$$

WOULD BE A SMOOTH ASSIGNMENT TO EACH $p \in S^n$ OF A TANGENT VECTOR TO $S^n$ AT $p$, I.E., A VECTOR FIELD ON $S^n$.

FOR $n=1$, $TS^1$ IS "JUST THE PRODUCT" $S^1 \times \mathbb{R}$ (SHORTLY WE WILL SAY THAT $TS^1$ IS "TRIVIAL").

FOR $n=2$, THE TOPOLOGY OF $TS^2$ IS MORE COMPLICATED (IN PARTICULAR, <u>NOT</u> HOMEOMORPHIC TO $S^2 \times \mathbb{R}^2$). THIS FOLLOWS FROM A RATHER DEEP THEOREM IN TOPOLOGY THAT SAYS EVERY CONTINUOUS VECTOR FIELD ON $S^2$ MUST VANISH SOMEWHERE.

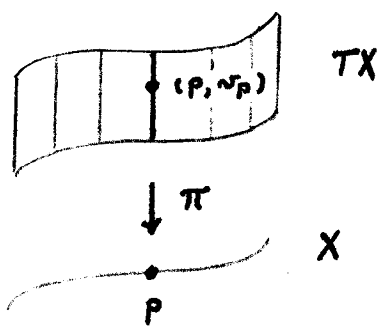
ODDLY ENOUGH, $TS^3$ <u>IS</u> HOMEOMORPHIC TO $S^3 \times \mathbb{R}^3$ AND $TS^7$ <u>IS</u> HOMEOMORPHIC TO $S^7 \times \mathbb{R}^7$, BUT THESE THREE ($n = 1, 3, 7$) ARE THE <u>ONLY</u> CASES IN WHICH THIS IS TRUE (THIS IS DEEP STUFF AND WE WILL NOT PROVE IT).

HOWEVER, EVEN THOUGH $TS^n$ IS USUALLY NOT HOMEOMORPHIC TO $S^n \times \mathbb{R}^n$, IT IS ALWAYS "LOCALLY HOMEOMORPHIC TO $S^n \times \mathbb{R}^n$" IN A SENSE WE WILL MAKE PRECISE.

WE NOW WANT TO DO FOR AN ARBITRARY SMOOTH MANIFOLD $X$ WHAT WE HAVE DONE FOR $S^n$, I.E., CONSTRUCT ITS "TANGENT BUNDLE" $TX$.

AS A <u>SET</u>,

$$TX = \{ (p, v_p) : p \in X, \; v_p \in T_p(X) \}$$




THERE IS A NATURAL <u>PROJECTION</u> $\pi : TX \to X$ GIVEN BY

$$\pi(p, v_p) = p \quad .$$

WE SUPPLY $TX$ WITH A TOPOLOGY AND A DIFFERENTIABLE STRUCTURE.

LET $(U, \varphi)$ BE A CHART ON $X$ WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$.
DEFINE

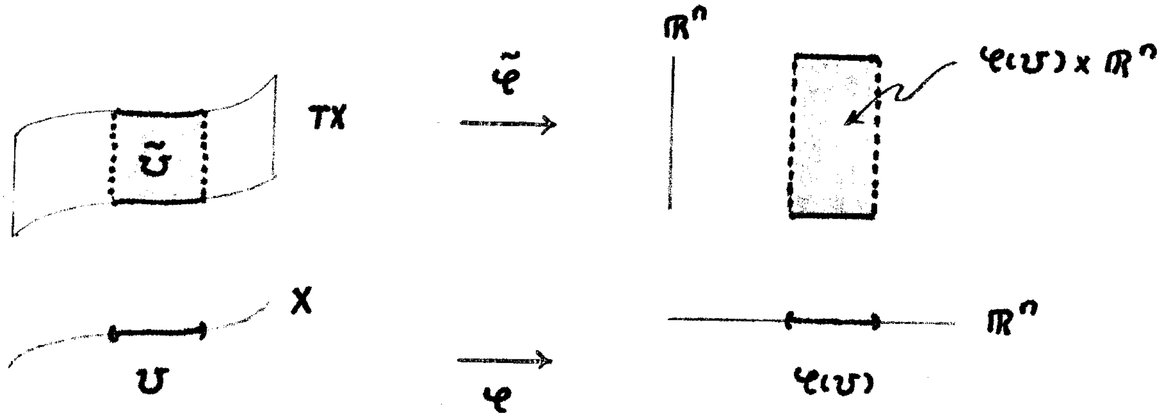
$$\tilde{U} = \pi^{-1}(U)$$

AND

$$\tilde{\varphi} : \tilde{U} \longrightarrow \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

$$\tilde{\varphi}(p, v_p) = (x^1(p), \dots, x^n(p), v_p(x^1), \dots, v_p(x^n))$$

<u>RECALL</u>: $v_p = v_p(x^1) \frac{\partial}{\partial x^1}\Big|_p + \dots + v_p(x^n) \frac{\partial}{\partial x^n}\Big|_p$




$\tilde{\varphi}$ IS A BIJECTION OF $\tilde{U}$ ONTO $\varphi(U) \times \mathbb{R}^n$ (AND WILL EVENTUALLY BE A CHART ON $TX$).

<u>TOPOLOGY</u>: DEFINE A SUBSET $\mathcal{U}$ OF $TX$ TO BE <u>OPEN</u> IF, FOR EVERY CHART $(U, \varphi)$ ON $X$,

$$\tilde{\varphi}(\tilde{U} \cap \mathcal{U})$$

IS OPEN IN $\mathbb{R}^{2n}$.

NEED TO SHOW THAT THIS COLLECTION OF SUBSETS $\mathcal{U}$ OF $TX$ ACTUALLY FORMS A TOPOLOGY FOR $TX$.

1. FOR ANY $(U,\varphi)$, $\tilde{\varphi}(\tilde{U} \cap \emptyset) = \tilde{\varphi}(\emptyset) = \emptyset$, WHICH IS OPEN IN $\mathbb{R}^{2n}$ SO $\emptyset$ IS OPEN IN $TX$.

2. FOR ANY $(U,\varphi)$, $\tilde{\varphi}(\tilde{U} \cap TX) = \tilde{\varphi}(\tilde{U}) = \varphi(U) \times \mathbb{R}^n$, WHICH IS OPEN IN $\mathbb{R}^{2n}$ SO $TX$ IS OPEN IN $TX$.

3. SUPPOSE $\mathcal{U}_\alpha$ IS OPEN IN $TX$ $\forall \alpha \in \mathcal{A}$, I.E., FOR ANY $(U,\varphi)$, $\tilde{\varphi}(\tilde{U} \cap \mathcal{U}_\alpha)$ IS OPEN IN $\mathbb{R}^{2n}$ $\forall \alpha \in \mathcal{A}$. THEN

$$\tilde{\varphi}\Big(\tilde{U} \cap \bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha\Big) = \tilde{\varphi}\Big(\bigcup_{\alpha \in \mathcal{A}} (\tilde{U} \cap \mathcal{U}_\alpha)\Big)$$

$$= \bigcup_{\alpha \in \mathcal{A}} \tilde{\varphi}(\tilde{U} \cap \mathcal{U}_\alpha) \quad \text{BECAUSE } \tilde{\varphi} \text{ IS A BIJECTION}$$

IS OPEN IN $\mathbb{R}^{2n}$ SO $\bigcup_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha$ IS OPEN IN $TX$.

4. FINITE INTERSECTIONS ARE HANDLED IN THE SAME WAY AS ARBITRARY UNIONS IN # 3.

<u>EXERCISE</u> : SHOW THAT, FOR ANY CHART $(V, \psi)$ ON $X$, $\tilde{V} = \pi^{-1}(V)$ IS OPEN IN $TX$.

<u>NOTE</u> : THIS IS NOT "OBVIOUS".

EVENTUALLY, THE $(\tilde{U}, \tilde{\varphi})$ WILL BE CHARTS ON $TX$ SO WE WILL NEED TO UNDERSTAND THEIR "OVERLAP MAPS". IT WILL BE CONVENIENT TO WRITE THESE OUT NOW.

LET $(U, \varphi)$ AND $(V, \psi)$ BE TWO CHARTS ON $X$ WITH $U \cap V \neq \phi$.

THEN

$$\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V) = \tilde{U} \cap \tilde{V}$$

SO

$$\tilde{U} \cap \tilde{V} \neq \phi .$$

MOREOVER,

$$\tilde{\varphi}(\tilde{U} \cap \tilde{V}) = \varphi(U \cap V) \times \mathbb{R}^n$$

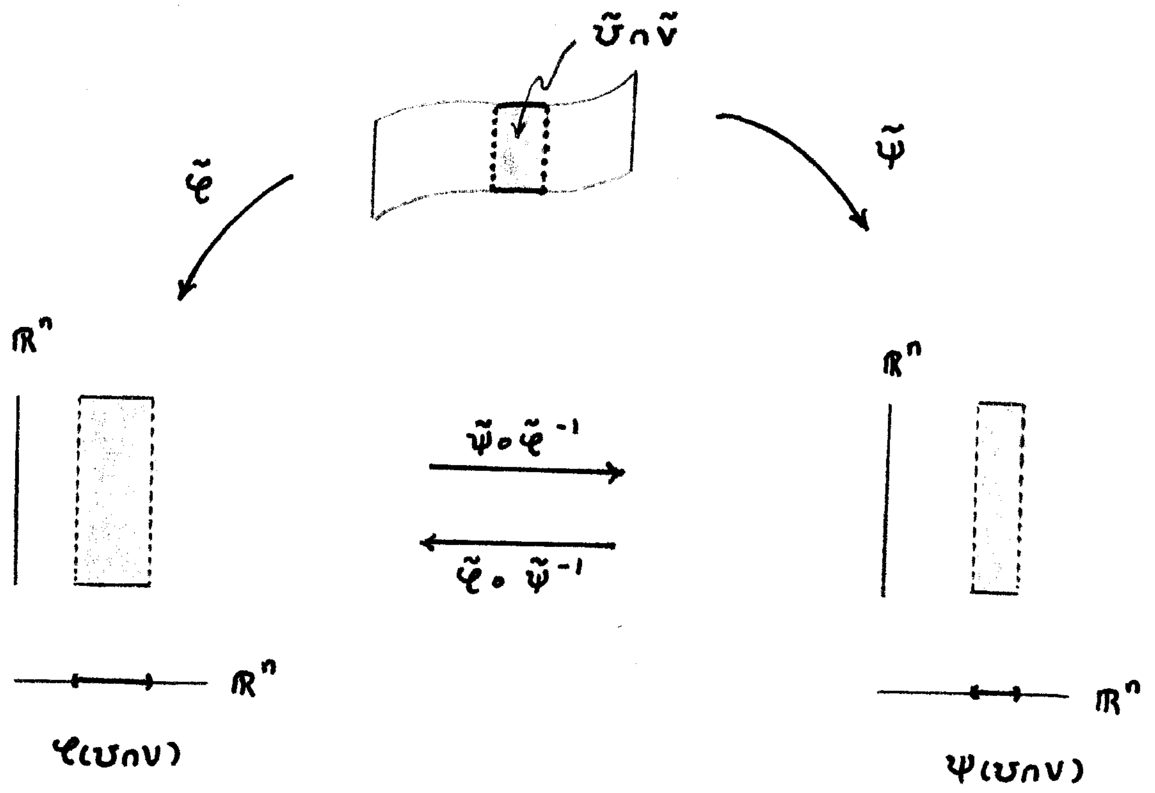
$$\tilde{\psi}(\tilde{U} \cap \tilde{V}) = \psi(U \cap V) \times \mathbb{R}^n$$

SO

$$\tilde{\varphi} \circ \tilde{\psi}^{-1} : \psi(U \cap V) \times \mathbb{R}^n \longrightarrow \varphi(U \cap V) \times \mathbb{R}^n$$

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

(AND $\tilde{\varphi} \circ \tilde{\psi}^{-1} = (\tilde{\psi} \circ \tilde{\varphi}^{-1})^{-1}$ )



WE'LL WRITE OUT $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ ( $\tilde{\varphi} \circ \tilde{\psi}^{-1}$ BEING SIMILAR ).

LET THE COORDINATE FUNCTIONS FOR $(U, \varphi)$ AND $(V, \psi)$ BE $x^1, \dots, x^n$ AND $y^1, \dots, y^n$, RESPECTIVELY.

FOR $(p, v_p) \in \tilde{U} \cap \tilde{V}$,

$$\tilde{\varphi}(p, v_p) = (x^1(p), \dots, x^n(p), v_p(x^1), \dots, v_p(x^n))$$
$$\tilde{\psi}(p, v_p) = (y^1(p), \dots, y^n(p), v_p(y^1), \dots, v_p(y^n))$$

AND, FOR ANY $(x^1, \dots, x^n, v^1, \dots, v^n) \in \varphi(U \cap V) \times \mathbb{R}^n$,

$$\tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) = (p, v_p)$$

WHERE

$$p = \varphi^{-1}(x^1, \dots, x^n) \text{ AND } v_p = v^1 \frac{\partial}{\partial x^1}\Big|_p + \dots + v^n \frac{\partial}{\partial x^n}\Big|_p .$$

THUS,

$$(\tilde{\psi} \circ \tilde{\varphi}^{-1})(x^1, \dots, x^n, v^1, \dots, v^n) = \tilde{\psi}(\tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n))$$
$$= \tilde{\psi}\left(\varphi^{-1}(x^1, \dots, x^n), v^i \frac{\partial}{\partial x^i}\Big|_p\right)$$
$$= \left(\psi(\varphi^{-1}(x^1, \dots, x^n)), \left(v^i \frac{\partial}{\partial x^i}\Big|_p\right)(y^1), \dots, \left(v^i \frac{\partial}{\partial x^i}\Big|_p\right)(y^n)\right)$$
$$= \left((\psi \circ \varphi^{-1})(x^1, \dots, x^n), v^i \frac{\partial y^1}{\partial x^i}(p), \dots, v^i \frac{\partial y^n}{\partial x^i}(p)\right) =$$
$$\left(y^1(x^1, \dots, x^n), \dots, y^n(x^1, \dots, x^n), \frac{\partial y^1}{\partial x^i}(\varphi^{-1}(x^1, \dots, x^n)) v^i, \dots, \frac{\partial y^n}{\partial x^i}(\varphi^{-1}(x^1, \dots, x^n)) v^i\right)$$

EACH COORDINATE FUNCTION OF WHICH IS A SMOOTH FUNCTION OF $x^1, \dots, x^n, v^1, \dots, v^n$.

CONCLUSION : $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ AND $\tilde{\varphi} \circ \tilde{\psi}^{-1}$ ARE INVERSE DIFFEOMORPHISMS OF THE OPEN SETS $\varphi(U \cap V) \times \mathbb{R}^n$ AND $\psi(U \cap V) \times \mathbb{R}^n$ IN $\mathbb{R}^{2n}$.

DON'T JUMP THE GUN, HOWEVER, BECAUSE WE STILL NEED TO SHOW THAT EACH

$$\tilde{\varphi} : \tilde{U} \longrightarrow \varphi(U) \times \mathbb{R}^n$$

IS A HOMEOMORPHISM. THIS WE DO NOW :

SINCE $\tilde{\varphi}$ IS A BIJECTION WE NEED ONLY SHOW THAT IT IS OPEN AND CONTINUOUS.

FOR THE FIRST OF THESE, LET $W$ BE AN OPEN SET IN $\tilde{U}$. SINCE $\tilde{U}$ IS OPEN IN $TX$, $W$ IS OPEN IN $TX$. BY DEFINITION, $\tilde{\varphi}(\tilde{U} \cap W) = \tilde{\varphi}(W)$ IS OPEN IN $\mathbb{R}^{2n}$. BUT $\tilde{\varphi}(W) \subseteq \varphi(U) \times \mathbb{R}^n$ SO $\tilde{\varphi}(W)$ IS OPEN IN $\varphi(U) \times \mathbb{R}^n$ AS REQUIRED.

TO PROVE CONTINUITY WE LET $Z$ BE OPEN IN $\varphi(U) \times \mathbb{R}^n$ AND SHOW THAT $\tilde{\varphi}^{-1}(Z)$ IS OPEN IN $\tilde{U}$, I.E., IN $TX$. FOR THIS WE MUST SHOW THAT, FOR ANY CHART $(V, \psi)$ ON $X$,

$$\tilde{\psi}(\tilde{V} \cap \tilde{\varphi}^{-1}(Z))$$

IS OPEN IN $\mathbb{R}^{2n}$. BUT

$$\tilde{\psi}(\tilde{V} \cap \tilde{\varphi}^{-1}(Z)) = \tilde{\psi}(\tilde{\varphi}^{-1}(Z) \cap (\tilde{U} \cap \tilde{V}))$$

SINCE $\tilde{\varphi}^{-1}(Z) \subseteq \tilde{U}$

$$= \tilde{\psi}(\tilde{\varphi}^{-1}(Z) \cap \tilde{\varphi}^{-1}(\varphi(U \cap V) \times \mathbb{R}^n))$$

SINCE $\tilde{\varphi}(\tilde{U} \cap \tilde{V}) = \varphi(U \cap V) \times \mathbb{R}^n$

$$= \tilde{\psi}(\tilde{\varphi}^{-1}(Z \cap (\varphi(U \cap V) \times \mathbb{R}^n)))$$

$$= (\tilde{\psi} \circ \tilde{\varphi}^{-1})(Z \cap (\varphi(U \cap V) \times \mathbb{R}^n))$$

WHICH IS OPEN BECAUSE $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ IS A HOMEOMORPHISM ON $\varphi(U \cap V) \times \mathbb{R}^n$.

THUS, WE HAVE DEFINED A TOPOLOGY ON TX AND PRODUCED A FAMILY OF CHARTS $(\tilde{U}, \tilde{\varphi})$ COVERING TX ANY TWO OF WHICH ARE $C^\infty$- RELATED.

BEFORE CONCLUDING THAT TX IS A SMOOTH MANIFOLD, HOWEVER, WE MUST SHOW THAT THE TOPOLOGY OF TX IS

1. HAUSDORFF
2. SECOND COUNTABLE.

<u>EXERCISE</u> : PROVE BOTH OF THESE.

THUS, TX IS A DIFFERENTIABLE MANIFOLD OF DIMENSION $2n$.

<u>EXERCISE</u>: PROVE THAT, WITH THE DIFFERENTIABLE STRUCTURE FOR $TX$ DESCRIBED ABOVE, THE NATURAL PROJECTION

$$\pi : TX \to X$$

$$\pi(p, v_p) = p$$

IS SMOOTH.

THERE IS ONE MORE ASPECT OF THE STRUCTURE OF $TX$ THAT IS CRUCIAL.

IT IS CALLED "LOCAL TRIVIALITY" AND WE NOW EXPLAIN WHAT THIS MEANS. CONSIDER THE COMPOSITION OF THE FOLLOWING MAPS:

$$\pi^{-1}(U) \xrightarrow{\tilde{\varphi}} \varphi(U) \times \mathbb{R}^n \xrightarrow{\varphi^{-1} \times id_{\mathbb{R}^n}} U \times \mathbb{R}^n$$

$$(p, v_p) \xrightarrow{\qquad \phi \qquad} (p, v_p(x^1), \ldots, v_p(x^n))$$

THIS IS A DIFFEOMORPHISM OF $\pi^{-1}(U)$ ONTO $U \times \mathbb{R}^n$ AND HAS THE PROPERTY THAT ITS RESTRICTION TO ANY

$$\pi^{-1}(p) \cong T_p(X)$$

IS AN ISOMORPHISM ONTO THE VECTOR SPACE

$$\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n .$$

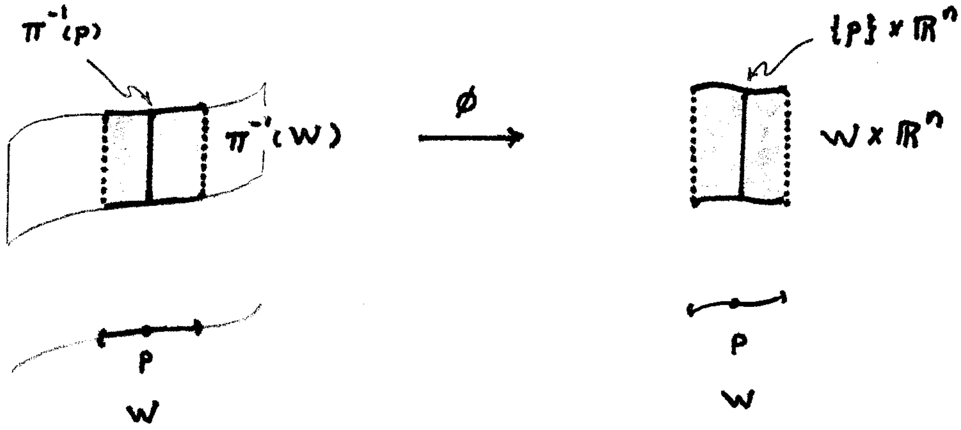
IN GENERAL, A <u>LOCAL TRIVIALIZATION</u> OF TX CONSISTS OF AN OPEN SET W IN X AND A DIFFEOMORPHISM

$$\phi : \pi^{-1}(W) \to W \times \mathbb{R}^n$$

OF $\pi^{-1}(W)$ ONTO $W \times \mathbb{R}^n$ WITH THE PROPERTY THAT, FOR EACH $p \in W$,

$$\phi \mid \pi^{-1}(p) : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$$

IS AN ISOMORPHISM OF VECTOR SPACES.




IF THERE IS SUCH A MAP DEFINED ON ALL OF $\pi^{-1}(X) = TX$, THEN IT IS CALLED A <u>GLOBAL TRIVIALIZATION</u> AND TX IS SAID TO BE <u>TRIVIAL</u>.

WITH ALL OF THE STRUCTURE WE HAVE JUST DESCRIBED, THE TRIPLE

$$(TX, \pi, X)$$

IS CALLED THE <u>TANGENT BUNDLE</u> OF X. FOR EACH $p \in X$, $\pi^{-1}(p)$ IS THE <u>FIBER</u> ABOVE p AND IS A VECTOR SPACE OF DIMENSION n (I.E., $T_p(X)$).

IF $W$ IS AN OPEN SUBSET OF $X$ AND

$$s : W \to TX$$

IS A SMOOTH MAP WITH THE PROPERTY THAT

$$\pi \circ s = id_W$$

($\pi(s(p)) = p$) IS CALLED A (<u>LOCAL</u>) <u>SECTION</u> OF $TX$. IF $W = X$, THEN $s$ IS CALLED A <u>GLOBAL</u> <u>SECTION</u> OF $TX$.

THUS, FOR EACH $p \in W$,

$$s(p) = (p, V(p))$$

FOR SOME

$$V(p) \in T_p(X).$$

A SECTION OF $TX$ IS ESSENTIALLY THE SAME THING AS A VECTOR FIELD

<u>EXERCISE</u>: SHOW THAT, WITH THE DIFFERENTIABLE STRUCTURE WE HAVE DEFINED ON $TX$, SMOOTHNESS OF THE SECTION $s$ IS EQUIVALENT TO OUR DEFINITION OF SMOOTHNESS FOR THE VECTOR FIELD $V$ (IF YOU NEED A HINT, SEE LEMMA 4.9, P. 37, OF GUDMUNDSSON.).

EVERY TX HAS AT LEAST ONE GLOBAL SECTION, I.E., THE <u>0-SECTION</u>

$$s_0 : X \to TX$$

$$s_0(p) = (p, 0_p)$$

WHERE $0_p \in T_p(X)$ IS THE ZERO ELEMENT OF THE VECTOR SPACE.

LOCALLY, THERE ARE LOTS OF SIMILAR "CONSTANT SECTIONS":

LET $(U, \varphi)$ BE A CHART ON $X$ AND $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ THE CORRESPONDING LOCAL TRIVIALIZATION OF TX. SELECT ANY $v_0 \in \mathbb{R}^n$ AND DEFINE

$$s_{v_0} : U \to TX$$

$$s_{v_0}(p) = \phi^{-1}(p, v_0)$$

<u>EXERCISE</u>: SHOW THAT THIS IS A (SMOOTH) LOCAL SECTION OF TX.

IN PARTICULAR, IF TX IS TRIVIAL THERE ARE LOTS OF GLOBAL SECTIONS THAT VANISH NOWHERE.

IF TX IS NOT TRIVIAL THIS IS A MUCH HARDER QUESTION AND WE WILL CONSIDER SOME EXAMPLES NEXT TIME.