WE NOW KNOW (ABSTRACTLY, AT LEAST) ABOUT THE TANGENT BUNDLE

$$\begin{tikzcd} TX \arrow[d, "\pi"] \\ X \end{tikzcd}$$

OF A SMOOTH MANIFOLD X AND ITS SECTIONS

$$\begin{tikzcd} & TX \arrow[d, "\pi"] \\ X \arrow[ur, "s"] \arrow[r, "id_X"'] & X \end{tikzcd}$$

BUT IT WOULD BE NICE TO SEE A FEW CONCRETE EXAMPLES.

TX IS GENERALLY QUITE DIFFICULT TO DESCRIBE EXPLICITLY, UNLESS IT IS TRIVIAL IN WHICH CASE IT IS DIFFEOMORPHIC TO $X \times \mathbb{R}^n$ ($n = \dim X$).

KNOWING WHETHER OR NOT THE TANGENT BUNDLE OF A GIVEN MANIFOLD X IS TRIVIAL IS, WELL, NONTRIVIAL. THERE ARE A NUMBER OF CONDITIONS KNOWN THAT GUARANTEE THIS, THE MOST USEFUL OF WHICH IS THE FOLLOWING:

<u>THEOREM</u>: THE TANGENT BUNDLE TX OF A SMOOTH n-MANIFOLD X IS TRIVIAL IF AND ONLY IF THERE EXISTS A SET

$$\{ s_1, \ldots, s_n \}$$

OF n SMOOTH SECTIONS

$$s_i : X \to TX \quad , \quad i = 1, \ldots, n,$$

WITH THE PROPERTY THAT, FOR EACH $p \in X$, THE VECTORS

$$\{ s_1(p), \ldots, s_n(p) \}$$

ARE LINEARLY INDEPENDENT IN THE FIBER $\pi^{-1}(p) \cong T_p(X)$.

<u>NOTE</u>: I WOULD LIKE TO LOOK AT SOME CONSEQUENCES OF THIS BEFORE GIVING THE PROOF, BUT YOU SHOULD NOTICE THAT THE IDEA IS QUITE SIMPLE:

IF TX IS TRIVIAL, THEN THERE IS A GLOBAL TRIVIALIZATION

$$\phi : \pi^{-1}(X) = TX \longrightarrow X \times \mathbb{R}^n$$

AND WE CAN GET OUR SECTIONS s_i BY DEFINING

$$s_i(p) = \phi^{-1}(p, (0, \ldots, \underset{\substack{\uparrow \\ i^{\text{TH}} \text{ SLOT}}}{1}, \ldots 0))$$

CONVERSELY, IF WE HAVE THE SECTIONS $s_1, \ldots, s_n$ WE CAN BUILD A GLOBAL TRIVIALIZATION BY DEFINING

$$\phi^{-1} : X \times \mathbb{R}^n \to TX$$

BY

$$\phi^{-1}(p, (v^1, \dots, v^n)) = (p, v^i s_i(p))$$

WE'LL GET BACK TO THE DETAILS SHORTLY. FIRST WE'LL DISCUSS THE FOLLOWING CONSEQUENCES:

1. $T\mathbb{R}^n$ IS TRIVIAL.
2. TS^1 IS TRIVIAL.
3. TS^2 IS NOT TRIVIAL.
4. TS^3 IS TRIVIAL.
5. IF G IS ANY LIE GROUP, THEN TG IS TRIVIAL.

FOR THE RECORD, IF TX IS TRIVIAL, THEN X IS SAID TO BE <u>PARALLELIZABLE</u>.

THE FIRST I'LL LEAVE FOR YOU.

<u>EXERCISE</u>: USE THE THEOREM ON PAGE 2 TO SHOW THAT $T\mathbb{R}^n$ IS TRIVIAL.

FOR S^1 (DIMENSION 1) THE THEOREM REQUIRES THAT WE EXHIBIT <u>ONE</u> SECTION OF TS^1 THAT IS NOWHERE ZERO, I.E., ONE NONVANISHING SMOOTH VECTOR FIELD ON S^1.

IT IS A SIMPLE MATTER TO JUST WRITE SUCH VECTOR FIELDS DOWN EXPLICITLY, BUT I WOULD LIKE TO DO IT IN A WAY THAT SUGGESTS A GENERALIZATION FOR S^3 AND FOR AN ARBITRARY LIE GROUP G.

S^1 IS A SUBMANIFOLD OF $\mathbb{R}^2 = \mathbb{C}$ AND THE TANGENT SPACE TO S^1 AT ANY $z \in S^1$ CAN BE IDENTIFIED WITH THE ORTHOGONAL COMPLEMENT OF THE VECTOR z IN $\mathbb{R}^2$. LET

$$e_0 = (1,0) \qquad (= 1 \in \mathbb{C})$$
$$e_1 = (0,1) \qquad (= i \in \mathbb{C})$$

BE THE ELEMENTS OF THE STANDARD BASIS FOR $\mathbb{R}^2$. NOTICE THAT

$$e_0 \in S^1$$

AND e_1 CAN BE REGARDED AS AN ELEMENT OF $T_{e_0}(S^1)$. FOR ANY $z = (x,y) \in S^1$

$$ze_0 = z = (x,y)$$
$$ze_1 = zi = (-y,x)$$

SO $ze_1 \in T_{ze_0}(S^1)$. THUS,

$$z \to (ze_0, ze_1)$$
$$(x,y) \to ((x,y), (-y,x))$$

IS A SMOOTH, NONZERO SECTION OF TS^1. THUS, TS^1 IS TRIVIAL. IN PARTICULAR,

$$TS^1 \cong S^1 \times \mathbb{R}$$

TO PRODUCE THIS NONVANISHING SECTION/VECTOR FIELD ON S^1 WE ESSENTIALLY TOOK A NONZERO TANGENT VECTOR e_1 AT THE IDENTITY ELEMENT e_0 AND USED COMPLEX MULTIPLICATION TO "ROTATE" e_1 TO ANY OTHER POINT $z \in S^1$.

NOW WE DO THE SAME THING FOR S^3 (THREE TIMES) USING QUATERNIONIC RATHER THAN COMPLEX MULTIPLICATION.

S^3 IS A SUBMANIFOLD OF $\mathbb{R}^4 = \mathbb{H}$ AND THE TANGENT SPACE TO S^3 AT ANY $q \in S^3$ CAN BE IDENTIFIED WITH THE ORTHOGONAL COMPLEMENT OF THE VECTOR q IN $\mathbb{R}^4$. LET

$$e_0 = (1,0,0,0) \qquad (= 1 \in \mathbb{H})$$
$$e_1 = (0,1,0,0) \qquad (= i \in \mathbb{H})$$
$$e_2 = (0,0,1,0) \qquad (= j \in \mathbb{H})$$
$$e_3 = (0,0,0,1) \qquad (= k \in \mathbb{H})$$

THEN

$$e_0 \in S^3$$

AND $e_1, e_2, e_3 \in T_{e_0}(S^3)$. FOR ANY $q = (x, y, u, v) \in S^3$

$$q e_0 = q = (x, y, u, v)$$
$$q e_1 = q i = (-y, x, -v, u)$$
$$q e_2 = q j = (-u, -v, x, y)$$
$$q e_3 = q k = (-v, u, -y, x)$$

THUS,

$$qe_1, qe_2, qe_3 \in T_{qe_0}(S^3)$$

SO

$$q \to (qe_0, qe_1) \quad , \quad (x,y,u,v) \to \big((x,y,u,v), (-y, x, -v, u)\big)$$

$$q \to (qe_0, qe_2) \quad , \quad (x,y,u,v) \to \big((x,y,u,v), (-u, -v, x, y)\big)$$

$$q \to (qe_0, qe_3) \quad , \quad (x,y,u,v) \to \big((x,y,u,v), (-v, u, -y, x)\big)$$

ARE THREE SMOOTH, NONZERO SECTIONS OF TS^3. AT EACH POINT $q = (x, y, u, v)$ THEY ARE LINEARLY INDEPENDENT BECAUSE IF $a, b, c \in \mathbb{R}$,

$$a(qe_1) + b(qe_2) + c(qe_3) = 0$$

$$\Rightarrow \quad q(ae_1 + be_2 + ce_3) = 0$$

$$\Rightarrow \quad ae_1 + be_2 + ce_3 = 0$$ (q HAS A MULTIPLICATIVE INVERSE IN $\mathbb{H}$)

$$\Rightarrow \quad a = b = c = 0$$

THUS, TS^3 IS TRIVIAL.

<u>EXERCISE</u>: FIND OUT WHAT THE CAYLEY NUMBERS ("OCTONIANS") ARE AND USE THEM TO PROVE THAT S^7 IS PARALLELIZABLE.

EXERCISE : SHOW THAT A SMOOTH, NONVANISHING VECTOR FIELD EXISTS ON ANY ODD DIMENSIONAL SPHERE S^{2n-1}.

THIS DOESN'T MAKE THEIR TANGENT BUNDLES TRIVIAL, HOWEVER.

S^1, S^3 AND S^7 ARE THE ONLY PARALLELIZABLE SPHERES

A MUCH DEEPER THEOREM FROM TOPOLOGY STATES THAT

ANY CONTINUOUS VECTOR FIELD ON AN EVEN DIMENSIONAL SPHERE S^{2n} MUST VANISH SOMEWHERE.

IN PARTICULAR, TS^{2n} IS NOT TRIVIAL.

WE WOULD LIKE TO USE THE SAME IDEA EMPLOYED FOR S^1 AND S^3 TO SHOW THAT ANY LIE GROUP IS PARALLELIZABLE

THE DETAILS ARE NOT AS SIMPLE THIS TIME BECAUSE WE DON'T HAVE SUCH AN EXPLICIT DESCRIPTION OF THE TANGENT SPACES, BUT THE IDEAS WE HAVE TO INTRODUCE ARE IMPORTANT FOR OTHER REASONS AS WELL.

FOR S^1 AND S^3 WE BEGAN WITH A NONZERO TANGENT VECTOR AT THE IDENTITY AND "PUSHED" IT TO NEW LOCATIONS USING THE GROUP OPERATION TO OBTAIN NONZERO VECTOR FIELDS.

THE "PUSHING" WAS EASY BECAUSE TANGENT VECTORS TO S^n CAN BE IDENTIFIED WITH POINTS IN $\mathbb{R}^{n+1}$ SO (WHEN $n = 1, 3, 7$) THEY CAN BE (COMPLEX, QUATERNION, CAYLEY) MULTIPLIED BY OTHER POINTS IN S^n.

NOW LET G BE AN ARBITRARY LIE GROUP WITH IDENTITY ELEMENT e AND CONSIDER THE TANGENT SPACE

$$T_e(G)$$

TO G AT e.

PICK ANY $V_e \in T_e(G)$. HOW DO WE "PUSH" V_e TO ANOTHER POINT $g \in G$?

SIMPLE ! LEFT TRANSLATION BY g

$$L_g : G \to G$$
$$L_g(h) = gh$$

IS A DIFFEOMORPHISM OF G ONTO G THAT CARRIES e TO g SO ITS DERIVATIVE AT e

$$(L_g)_{*e} : T_e(G) \longrightarrow T_g(G)$$

CARRIES $T_e(G)$ ISOMORPHICALLY ONTO $T_g(G)$ SO WE MAY "PUSH" V_e BY $(L_g)_{*e}$ TO

$$V_g := (L_g)_{*e}(V_e).$$

NOTICE THAT IF $V_e \neq 0$, THEN $V_g \neq 0$ ($(L_g)_{*e}$ IS AN ISOMORPHISM) SO WE HAVE A NONZERO VECTOR FIELD

$$g \longrightarrow V_g$$

ON G

THE ONLY PROBLEM IS THAT WE DO NOT KNOW (YET) THAT THIS VECTOR FIELD IS SMOOTH.

<u>NOTE</u>: IF WE PROVE SMOOTHNESS WE WILL BE ABLE TO CONCLUDE THAT TG IS TRIVIAL. TO SEE THIS, CHOOSE A BASIS $\{V_1, \ldots, V_n\}$ FOR $T_e(G)$ AND, FOR EACH $i = 1, \ldots, n$, CONSIDER THE CORRESPONDING (SMOOTH) VECTOR FIELD

$$g \longrightarrow (L_g)_{*e}(V_i)$$

ON G. THESE ARE LINEARLY INDEPENDENT AT EACH $g \in G$ SO TG IS TRIVIAL.

SO, WE'LL PROVE SMOOTHNESS. LET'S FORMULATE A GENERAL DEFINITION:

A VECTOR FIELD V (NOT NECESSARILY SMOOTH) ON A LIE GROUP G IS SAID TO BE <u>LEFT-INVARIANT</u> IF, FOR EVERY g IN G,

$$V(g) = (L_g)_{*e}(V(e))$$

<u>EXERCISE</u>: SHOW THAT THIS IS THE CASE IF AND ONLY IF, FOR ALL g AND h IN G,

$$V(gh) = (L_g)_{*h}(V(h))$$

<u>THEOREM</u>: ANY LEFT-INVARIANT VECTOR FIELD V ON A LIE GROUP G IS NECESSARILY SMOOTH.

<u>PROOF</u>: BEGIN WITH A FEW OBSERVATIONS:

1. IT'S ENOUGH TO PROVE THAT V IS SMOOTH ON SOME NEIGHBORHOOD OF e IN G.

TO SEE THIS, SUPPOSE $(\mathcal{U}, \varphi)$ IS A CHART FOR G AT e WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$ AND SUPPOSE WE HAVE SHOWN THAT V IS SMOOTH ON $\mathcal{U}$.

SINCE

$$V = V(x^i)\frac{\partial}{\partial x^i}$$

ON U, THE REAL-VALUED FUNCTIONS $V(x^i)$ ARE C^∞ ON U.

FOR ANY OTHER $g \in G$, L_g IS A DIFFEOMORPHISM SO

$$(L_g(U),\ \varphi \circ L_{g^{-1}})$$

IS A CHART AT g. LET THE COORDINATE FUNCTIONS BE $y^1, \dots, y^n$. THEN

$$y^i = x^i \circ L_{g^{-1}} \quad , \; i = 1, \dots, n .$$

ON $L_g(U)$ WE HAVE

$$V = V(y^i)\frac{\partial}{\partial x^i}$$

BUT

$$V(y^i) = V(x^i \circ L_{g^{-1}})$$

AND THESE ARE SMOOTH ON $L_g(U)$ BECAUSE

$$V(x^i \circ L_{g^{-1}}) = V(x^i) \circ L_{g^{-1}}$$

<u>EXERCISE</u> : PROVE THIS. <u>HINT</u> : EVALUATE THE LEFT-HAND SIDE AT $a \in L_g(U)$, RECALL THE DEFINITION OF $(L_{g^{-1}})_{*a}$ AND USE LEFT-INVARIANCE OF V.

THUS, SMOOTHNESS OF V ON U IMPLIES ITS SMOOTHNESS ON A NEIGHBORHOOD OF ANY POINT g IN G.

2. LET (U, φ) BE A CHART AT e WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$. THEN WE CAN FIND AN OPEN NEIGHBORHOOD U' OF e SUCH THAT

(i) $a, b \in U' \Rightarrow ab \in U'$

(ii) $U' \subseteq U$

(iii) $a \in U' \Rightarrow L_a(U') \subseteq U$

HERE'S THE REASON: THE GROUP MULTIPLICATION

$$G \times G \xrightarrow{m} G$$
$$m(a,b) = ab$$

IS SMOOTH (AND THEREFORE CONTINUOUS) SO

$$m^{-1}(U) = \{(a,b) \in G \times G : ab \in U\}$$

IS OPEN IN $G \times G$. $(e,e) \in m^{-1}(U)$ SO THERE IS A BASIC OPEN SET $U_1 \times U_2$ WITH

$$(e,e) \in U_1 \times U_2 \subseteq m^{-1}(U)$$

IF $U' = U_1 \cap U_2$, THEN $(e,e) \in U' \times U' \subseteq m^{-1}(U)$ SO (i) IS SATISFIED. (ii) IS SATISFIED BECAUSE $e \in U'$ IMPLIES $a = ae \in U$ FOR ALL $a \in U'$. SIMILARLY FOR (iii).

NOW WE BEGIN THE PROOF.

ASSUME V SATISFIES $V(g) = (L_g)_{*e}(V(e))$ FOR EVERY $g \in G$.

LET (U, φ) BE A CHART AT $e \in G$ WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$.

BY # 2 ABOVE WE CAN ASSUME

$$a, b \in U \Rightarrow ab \in U$$

(SO, IN PARTICULAR, $L_a(U) \subseteq U$ FOR EVERY $a \in U$).

ON U WE HAVE

$$V = V(x^i)\frac{\partial}{\partial x^i}$$

AND WE MUST SHOW THAT $V(x^i)$ IS C^∞ FOR EVERY $i = 1, \ldots, n$.

FOR EACH $a \in U$,

$$V(x^i)(a) = V(a)(x^i) = ((L_a)_{*e}(V(e)))(x^i)$$

$$= V(e)(x^i \circ L_a)$$

LET

$$V(e) = v^j \frac{\partial}{\partial x^j}\Big|_e$$

THEN

$$V(x^i)(a) = \left(v^j \frac{\partial}{\partial x^j}\Big|_e\right)(x^i \circ L_a)$$

$$= v^j \frac{\partial}{\partial x^j}\Big|_e (x^i \circ L_a)$$

$$= v^j \underbrace{D_j(x^i \circ L_a \circ \varphi^{-1})(\varphi(e))}_{\substack{\text{NEED TO SHOW THAT} \\ \text{THIS IS A SMOOTH} \\ \text{FUNCTION OF } a \in U}}$$

WE'LL COMPUTE $x^i \circ L_a$:

FOR EACH $b \in U$,

$$(x^i \circ L_a)(b) = x^i(L_a(b))$$

$$= x^i(ab)$$

NOW NOTICE THAT THIS IS SMOOTH AS A FUNCTION OF $(a,b) \in U \times U$ BECAUSE

$$(a,b) \in U \times U \xrightarrow{m} ab \in U \xrightarrow{\varphi} \varphi(ab) = (x^1(ab), \dots, x^n(ab))$$

IS SMOOTH ON $U \times U$.

IN PARTICULAR,

$$(a,b) \in U \times U \xrightarrow{x^i \circ m} x^i(ab) = (x^i \circ L_a)(b)$$

IS SMOOTH ON $U \times U$.

THUS, IT'S COORDINATE EXPRESSION RELATIVE TO THE CHART $(U \times U, \varphi \times \varphi)$ IS SMOOTH :

$$\begin{tikzcd} U \times U \arrow[r, "x^i \circ m"] \arrow[d, "\varphi \times \varphi"'] & \mathbb{R} \\ \varphi(U) \times \varphi(U) \arrow[ur, dotted, "(x^i \circ m) \circ (\varphi \times \varphi)^{-1}"'] & \end{tikzcd}$$

WRITE

$$(\varphi \times \varphi)(a,b) = (\varphi(a), \varphi(b)) = (x^1(a), \ldots, x^n(a), x^1(b), \ldots, x^n(b))$$
$$= (a^1, \ldots, a^n, b^1, \ldots, b^n)$$

THEN

$$(x^i \circ m) \circ (\varphi \times \varphi)^{-1}(a^1, \ldots, a^n, b^1, \ldots, b^n)$$

$$= (x^i \circ m)(\varphi^{-1}(a^1, \ldots, a^n), \varphi^{-1}(b^1, \ldots, b^n))$$

$$= x^i(\varphi'(a^1, \ldots, a^n)\varphi^{-1}(b^1, \ldots, b^n))$$

IS A SMOOTH FUNCTION OF $a^1, \ldots, a^n, b^1, \ldots, b^n$.

NOW COMPUTE $D_j(x^i \circ L_a \circ \varphi^{-1})$:

$$(x^i \circ L_a \circ \varphi^{-1})(b^1,\dots,b^n) = x^i(a\varphi^{-1}(b^1,\dots,b^n))$$

$$D_j(x^i \circ L_a \circ \varphi^{-1})(b^1,\dots,b^n) = \frac{\partial}{\partial b^j} x^i(a\varphi^{-1}(b^1,\dots,b^n))$$

IS A SMOOTH FUNCTION OF $a^1,\dots,a^n, b^1,\dots,b^n$

THUS,

$$D_j(x^i \circ L_a \circ \varphi^{-1})(\varphi(e)) = \frac{\partial}{\partial b^j} x^i(a\varphi^{-1}(b^1,\dots,b^n))(\varphi(e))$$

IS A SMOOTH FUNCTION OF $a^1,\dots,a^n$

AND THIS IS WHAT WE WERE SUPPOSED TO SHOW. $\square$

THIS COMPLETES THE PROOF THAT LEFT INVARIANT VECTOR FIELDS ON A LIE GROUP G ARE SMOOTH AND THEREFORE ALSO THAT

<u>ANY LIE GROUP IS PARALLELIZABLE</u>.

MUCH MORE CAN BE DONE WITH THESE LEFT INVARIANT VECTOR FIELDS, HOWEVER.

<u>NOTE</u>: V, W LEFT INVARIANT AND $a, b \in \mathbb{R} \Rightarrow aV + bW$ LEFT INVARIANT (WHY?)

SET OF ALL LEFT INVARIANT VECTOR FIELDS ON G IS THEREFORE A REAL VECTOR SPACE.

THE MAP

$$V \longrightarrow V(e)$$

CARRIES THIS VECTOR SPACE ISOMORPHICALLY ONTO $T_e(G)$ (WHY?)

MOREOVER,

<u>CLAIM</u>: V, W LEFT INVARIANT $\Rightarrow$ LIE BRACKET $[V, W]$ IS LEFT INVARIANT

<u>RECALL</u>: $[V, W]_p(f) = V_p(Wf) - W_p(Vf)$

<u>PROOF</u>: ASSUME $V_g = (L_g)_{*e}(V_e)$ AND $W_g = (L_g)_{*e}(W_e)$ FOR ALL $g \in G$. THEN

$$\begin{aligned}
[V,W]_g(f) &= V_g(Wf) - W_g(Vf) \\
&= ((L_g)_{*e}(V_e))(Wf) - ((L_g)_{*e}(W_e))(Vf) \\
&= V_e(Wf \circ L_g) - W_e(Vf \circ L_g) \\
&= V_e(W(f \circ L_g)) - W_e(V(f \circ L_g))
\end{aligned}$$

<u>EXERCISE</u>: VERIFY THAT $Wf \circ L_g = W(f \circ L_g)$

AND $Vf \circ L_g = V(f \circ L_g)$

THUS,

$$[V,W]_g(f) = [V,W]_e(f \circ L_g)$$

$$= (L_g)_{*e}([V,W]_e)(f)$$

SO

$$[V,W]_g = (L_g)_{*e}([V,W]_e)$$

AS REQUIRED. $\square$

<u>CONCLUSION</u>: THE COLLECTION OF ALL LEFT INVARIANT VECTOR FIELDS ON A LIE GROUP G FORMS A LIE ALGEBRA $\mathfrak{g}$ UNDER LIE BRACKET, CALLED THE <u>LIE ALGEBRA OF</u> G.

AS A VECTOR SPACE, $\mathfrak{g}$ IS ISOMORPHIC TO $T_e(G)$. WE SHOW NOW THAT, WHEN G IS ONE OF THE CLASSICAL LIE GROUPS, SO THAT $T_e(G)$ IS IDENTIFIED WITH A SPACE OF MATRICES, $[V,W]_e$ IS JUST THE <u>MATRIX COMMUTATOR</u> OF V_e AND W_e.

THUS, OUR "OLD" DEFINITIONS OF THE LIE ALGEBRAS OF $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $O(n)$, $SO(n)$, $SL(n,\mathbb{R})$, $U(n)$ AND $SU(n)$ AGREE WITH THE "NEW" DEFINITION IN THE SENSE THAT $V \to V_e$ IS A LIE ALGEBRA ISOMORPHISM.

I WILL PROVE THIS ONLY FOR $GL(n,\mathbb{R})$. THE COMPLEX CASE $GL(n,\mathbb{C})$ IS SIMILAR. ALL OF THE REMAINING CASES FOLLOW FROM THE FACT THAT $O(n)$, $SO(n)$, ... ARE SUBGROUPS (AND SUBMANIFOLDS) OF THESE.

LET $G = GL(n,\mathbb{R})$ AND $e = id$. THEN $GL(n,\mathbb{R})$ IS AN OPEN SUBMANIFOLD OF $\mathbb{R}^{n^2}$ AND $T_{id}(GL(n,\mathbb{R}))$ IS ISOMORPHIC TO ALL OF $\mathbb{R}^{n^2}$.

LET x^{ij}, $i,j = 1,\dots,n$, BE THE STANDARD COORDINATE (I.E., ENTRY) FUNCTIONS ON $\mathbb{R}^{n^2}$.

IDENTIFY ANY $n \times n$ MATRIX $A = (A^{ij})$ WITH THE TANGENT VECTOR

$$A = A^{ij} \frac{\partial}{\partial x^{ij}} \Big|_{id}$$

LET $\underline{A}$ BE THE UNIQUE LEFT INVARIANT VECTOR FIELD ON $GL(n,\mathbb{R})$ WITH

$$\underline{A}(id) = A$$

<u>OBJECTIVE</u> : FOR TWO SUCH,

$$[\underline{A},\underline{B}]_{id} = [A,B] = AB - BA$$

WE COMPUTE THE COMPONENT FUNCTIONS $\underline{A}x^{kl}$ OF $\underline{A}$ AS FOLLOWS: FOR ANY $g \in GL(n,\mathbb{R})$,

$$(\underline{A}x^{kl})(g) = \underline{A}_g(x^{kl}) = ((L_g)_{*id}(A))(x^{kl})$$
$$= A(x^{kl} \circ L_g)$$

NOW, $x^{kl} \circ L_g : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$ IS GIVEN BY

$$(x^{kl} \circ L_g)(h) = x^{kl}(gh)$$

$= kl$- ENTRY IN THE MATRIX PRODUCT gh

$$= \sum_{\alpha=1}^{n} g^{k\alpha} h^{\alpha l} = \sum_{\alpha=1}^{n} g^{k\alpha} x^{\alpha l}(h)$$

FOR ANY $h \in GL(n,\mathbb{R})$.

THUS, FOR EACH FIXED $g \in GL(n,\mathbb{R})$,

$$x^{kl} \circ L_g$$

IS A LINEAR FUNCTION OF THE COORDINATES IN $GL(n,\mathbb{R})$ AND

$$\frac{\partial}{\partial x^{ij}}(x^{kl} \circ L_g) = \begin{cases} 0, & j \neq l \\ g^{ki}, & j = l \end{cases}$$

THUS,

$$(\underline{A}x^{kl})(g) = A(x^{kl}\circ L_g) = \left(A^{ij}\frac{\partial}{\partial x^{ij}}\Big|_{id}\right)(x^{kl}\circ L_g)$$

$$= \sum_{i=1}^{n} A^{il} g^{ki}$$

$$= \sum_{i=1}^{n} g^{ki} A^{il} \qquad = \sum_{i=1}^{n} x^{ki}(g)\, A^{il}$$

$$= kl\text{-ENTRY IN THE MATRIX PRODUCT } gA$$

WE CONCLUDE THAT

$$\underline{A}(g) = (\underline{A}x^{kl})(g)\,\frac{\partial}{\partial x^{kl}}\Big|_g \quad = (gA)^{kl}\,\frac{\partial}{\partial x^{kl}}\Big|_g$$

AGAIN IDENTIFYING TANGENT VECTORS TO POINTS IN $GL(n,\mathbb{R})$ WITH MATRICES WE WRITE THIS

$$\underline{A}(g) = gA.$$

FROM

$$\underline{A}x^{kl} = \sum_{i=1}^{n} x^{ki} A^{il}$$

WE FIND

$$\frac{\partial}{\partial x^{ij}}(\underline{A}x^{kl}) = \begin{cases} 0 & , k \neq i \\ A^{jl} & , k = i \end{cases}$$

NOW SUPPOSE $B = (B^{ij})$ IS ANOTHER REAL $n \times n$ MATRIX AND IDENTIFY B WITH THE TANGENT VECTOR

$$\underline{B} = B^{ij} \frac{\partial}{\partial x^{ij}}\Big|_{id} .$$

THEN

$$\underline{B}(\underline{A}x^{kl}) = B^{ij} \frac{\partial}{\partial x^{ij}}\Big|_{id} (\underline{A}x^{kl})$$
$$= \sum_{j=1}^{n} B^{kj} A^{jl}$$
$$= kl\text{-ENTRY OF } BA$$

SWITCHING A AND B GIVES

$$\underline{A}(\underline{B}x^{kl}) = \sum_{j=1}^{n} A^{kj} B^{jl}$$
$$= kl\text{-ENTRY OF } AB$$

THUS,

$$[\underline{A}, \underline{B}]_{id}(x^{kl}) = \underline{A}(\underline{B}x^{kl}) - \underline{B}(\underline{A}x^{kl})$$
$$= \sum_{j=1}^{n} (A^{kj} B^{jl} - B^{kj} A^{jl})$$
$$= kl\text{-ENTRY OF } AB - BA = [A, B]$$

$$[\underline{A}, \underline{B}]_{id} = [A, B]$$

AS REQUIRED.

ALTHOUGH WE WILL PROBABLY HAVE NO TIME FOR SUCH THINGS I WANTED TO MENTION A GENERALIZATION OF OUR CONSTRUCTION OF THE TANGENT BUNDLE THAT PLAYS A FUNDAMENTAL ROLE IN MODERN DIFFERENTIAL GEOMETRY.

LET $\mathbb{F}$ DENOTE EITHER OF THE FIELDS $\mathbb{R}$ OR $\mathbb{C}$. LET k BE A POSITIVE INTEGER. A <u>SMOOTH</u> <u>VECTOR</u> <u>BUNDLE</u> OF <u>FIBER</u> <u>DIMENSION</u> k OVER $\mathbb{F}$ IS A TRIPLE

$$(E, \pi, X)$$

WHERE E AND X ARE SMOOTH MANIFOLDS AND

$$\pi : E \to X$$

IS A SMOOTH MAP OF E ONTO X SUCH THAT

(1) FOR EACH $p \in X$ THE <u>FIBER</u> $\pi^{-1}(p)$ OVER p HAS THE STRUCTURE OF A k-DIMENSIONAL VECTOR SPACE OVER $\mathbb{F}$.

(2) FOR EACH $p \in X$ THERE IS AN OPEN NEIGHBORHOOD U OF p IN X AND A DIFFEOMORPHISM

$$\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{F}^k$$

SUCH THAT, FOR EACH $x \in U$,

$$\phi|\pi^{-1}(x) \quad : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{F}^k$$

IS AN ISOMORPHISM OF THE VECTOR SPACE $\pi^{-1}(x)$ ONTO THE VECTOR SPACE $\{x\} \times \mathbb{F}^k$.