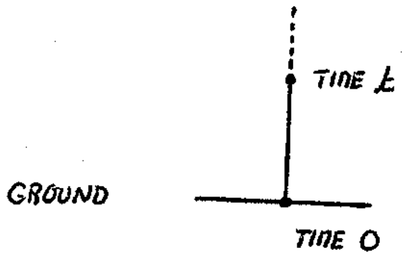
TANGENT LINES AND RATES OF CHANGE

HERE'S A FACT FROM PHYSICS : A BALL IS THROWN VERTICALLY UPWARD FROM THE GROUND AT AN INITIAL VELOCITY OF 64 FT/SEC. THEN t SEC LATER ITS HEIGHT S IN FEET IS GIVEN BY



$$S(t) = -16t^2 + 64t$$

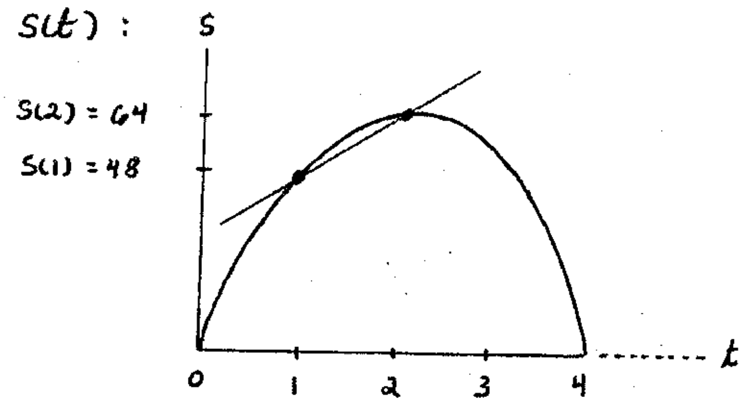
E.G., $S(1) = -16\cdot1^2 + 64\cdot1 = 48$

$S(2) = -16\cdot2^2 + 64\cdot2 = 64$

$S(3) = -16\cdot3^2 + 64\cdot3 = 48$

$S(4) = -16\cdot4^2 + 64\cdot4 = 0$

GRAPH OF $S(t)$:



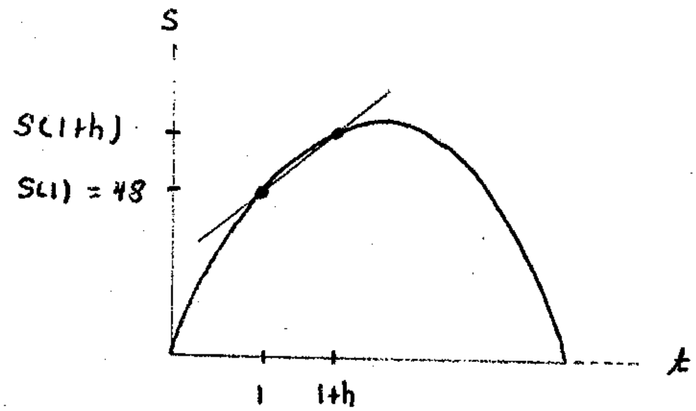
<u>NOTE</u> : <u>AVERAGE</u> <u>VELOCITY</u> OF THE BALL OVER THE TIME INTERVAL FROM $t=1$ TO $t=2$ IS

$$\frac{\text{DISTANCE TRAVELED}}{\text{LENGTH OF TIME INTERVAL}} = \frac{S(2)-S(1)}{2-1} = \frac{64-48}{1} = 16 \text{ FT/SEC}$$

WHICH IS ALSO THE <u>SLOPE</u> OF THE STRAIGHT LINE JOINING THE POINTS (1, 48) AND (2, 64) ON THE GRAPH OF $S(t)$.

SIMILARLY, THE AVERAGE VELOCITY OF THE BALL BETWEEN $t=2$ AND $t=3$ IS -16 FT/SEC AND THIS IS THE SLOPE OF THE LINE JOINING $(2,64)$ AND $(3,48)$.

NEXT, LET'S COMPUTE THE AVERAGE VELOCITY OF THE BALL OVER SOME "SMALL" TIME INTERVAL FROM $t=1$ TO $t=1+h$.



$$\frac{s(1+h)-s(1)}{h} = \frac{-16(1+h)^2+64(1+h) - 48}{h}$$

$$= \frac{-16(1+2h+h^2)+64+64h-48}{h}$$

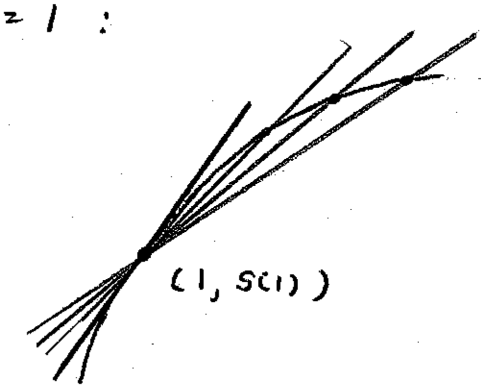
$$= \frac{-16-32h-16h^2+64+64h-48}{h}$$

$$= \frac{-16h^2+32h}{h} = -16h+32$$

= SLOPE OF THE LINE JOINING THE POINTS $(1, s(1))$ AND $(1+h, s(1+h))$ ON THE GRAPH OF $s(t)$.

THIS IS TRUE FOR ANY (NONZERO) h. IF h IS VERY SMALL, THE BALL HAS VERY LITTLE TIME TO SLOW DOWN OR SPEED UP AND THIS AVERAGE VELOCITY IS VERY CLOSE TO THE BALL'S VELOCITY " AT $t = 1$ " (" SPEEDOMETER READING " , " INSTANTANEOUS VELOCITY ")

AS $h \to 0$ THE AVERAGE VELOCITY APPROACHES THE INSTANTANEOUS VELOCITY AT $t = 1$:



<u>INSTANTANEOUS</u> <u>VELOCITY</u> AT $t = 1$

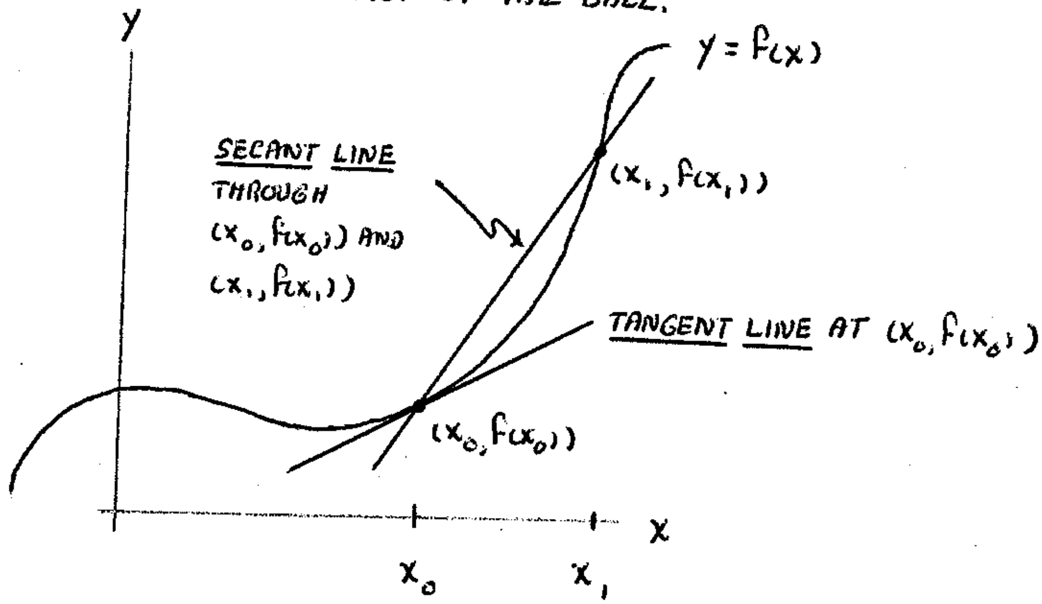
$$= \lim_{h \to 0} \frac{s(1+h) - s(1)}{h}$$

$$= \lim_{h \to 0} (-16h + 32)$$

$$= 32$$

= <u>SLOPE</u> OF THE <u>TANGENT</u> <u>LINE</u> (1, s(1))

NOW WE WILL DO FOR AN ARBITRARY FUNCTION $f(x)$ WHAT WE JUST DID FOR THE POSITION FUNCTION OF THE BALL.



<u>AVERAGE RATE OF CHANGE OF</u> $f(x)$ <u>OVER THE INTERVAL</u> $[x_0, x_1]$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

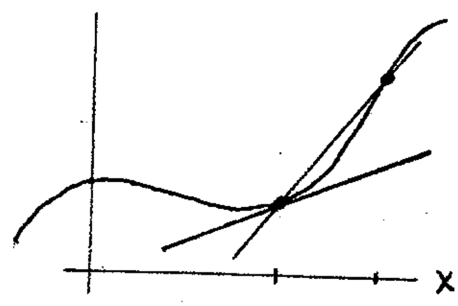
= <u>SLOPE OF SECANT LINE</u> THROUGH $(x_0, f(x_0))$ AND $(x_1, f(x_1))$

<u>INSTANTANEOUS RATE OF CHANGE OF</u> $f(x)$ <u>AT</u> x_0

$$= \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$ (PROVIDED THE LIMIT EXISTS)

= <u>SLOPE OF TANGENT LINE</u> AT $(x_0, f(x_0))$

SOMETIMES IT IS MORE CONVENIENT TO USE THE FOLLOWING EQUIVALENT FORMULA :



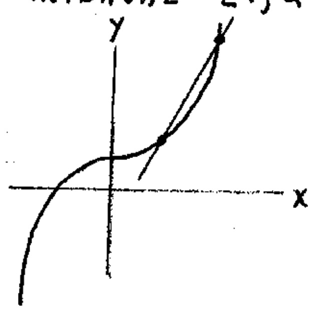
INSTANTANEOUS RATE OF CHANGE OF $f(x)$ AT x_0

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(PROVIDED THE LIMIT EXISTS)}$$

$= $ SLOPE OF TANGENT LINE AT $(x_0, f(x_0))$

EXAMPLES :

1. FIND THE AVERAGE RATE OF CHANGE OF $f(x) = x^3 + 1$ OVER THE INTERVAL $[1, 2]$



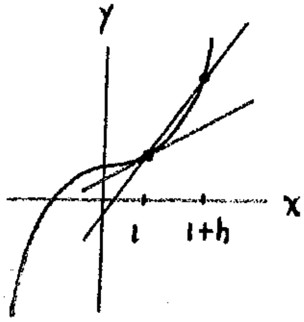
$[x_0, x_1] = [1, 2]$

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{(x_1^3+1) - (x_0^3+1)}{x_1 - x_0}$$

$$= \frac{(2^3+1) - (1^3+1)}{2-1}$$

$$= 7$$

2. FIND THE INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3 + 1$ AT $x_0 = 1$.



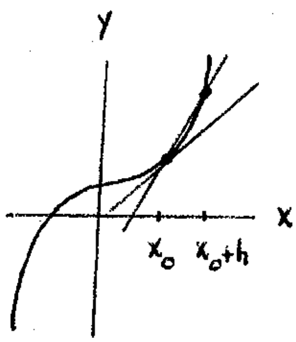
$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{((1+h)^3+1) - (1^3+1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+3h+3h^2+h^3+1) - 2}{h}$$

$$= \lim_{h\to 0} \frac{h(3+3h+h^2)}{h} = \lim_{h\to 0} (3+3h+h^2) = 3$$

3. FIND THE INSTANTANEOUS RATE OF CHANGE OF $f(x) = x^3 + 1$ AT x_0.



$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{((x_0+h)^3+1) - (x_0^3+1)}{h}$$

$$= \lim_{h\to 0} \frac{x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 + 1 - x_0^3 - 1}{h}$$

$$= \lim_{h\to 0} \frac{h(3x_0^2 + 3x_0h + h^2)}{h}$$

$$= \lim_{h\to 0} (3x_0^2 + 3x_0h + h^2)$$

$$= 3x_0^2$$