A smooth surface $\Sigma$ in $\mathbb{R}^3$ has, at each point $p \in \Sigma$, a tangent plane $T_p(\Sigma)$ consisting of all of the velocity vectors to all of the smooth curves in $\Sigma$ through $p$.

The velocity vectors are thought of as ordinary vectors in $\mathbb{R}^3$.

Similarly, a differentiable manifold $\mathcal{X}$ has, at each point $p \in \mathcal{X}$, a "tangent space" $T_p(\mathcal{X})$, but there is no "ambient Euclidean space" like $\mathbb{R}^3$ in which to view its elements as living so a different definition is required.

Motivation: Return to the smooth surface $\Sigma$ in $\mathbb{R}^3$, some point $p \in \Sigma$ and some tangent vector $\alpha_p \in T_p(\Sigma)$. Any smooth real-valued function $f \in C^\infty(\Sigma)$ has a derivative at $p$ with respect to $\alpha_p$ given by

$$\nabla f(p) \cdot \alpha_p$$

(in calculus, when $\alpha_p$ is a unit vector this is called the directional derivative of $f$ at $p$ in the direction $\alpha_p$). Thus, $\alpha_p$ gives rise to a mapping from $C^\infty(\Sigma)$ to $\mathbb{R}$ which assigns to each $f \in C^\infty(\Sigma)$ this number. Use the same symbol to denote this mapping:

$$\alpha_p : C^\infty(\Sigma) \rightarrow \mathbb{R}$$

$$\alpha_p(f) = \nabla f(p) \cdot \alpha_p \quad \forall f \in C^\infty(\Sigma)$$

Properties:

(a) Linearity: $\alpha_p(af + bg) = a\alpha_p(f) + b\alpha_p(g)$ $\forall a, b \in \mathbb{R}$ $\forall f, g \in C^\infty(\Sigma)$

(b) Leibnitz product rule: $\alpha_p(fg) = f(p)\alpha_p(g) + \alpha_p(f)g(p)$ $\forall f, g \in C^\infty(\Sigma)$. 
IT IS NOT OBVIOUS, BUT ONE CAN PROVE THAT ANY MAPPING $C^\infty(\Sigma) \to \mathbb{R}$ SATISFYING THESE TWO PROPERTIES MUST BE $f \mapsto \nabla^p_f \cdot \nabla$ FOR A UNIQUE $\nabla \in T_p(\Sigma)$.

THUS, WE CAN COMPLETELY IDENTIFY ELEMENTS OF $T_p(\Sigma)$ WITH MAPS $C^\infty(\Sigma) \to \mathbb{R}$ SATISFYING (a) AND (b). THIS THEN GIVES US A WAY OF DEFINING TANGENT VECTORS TO ARBITRARY MANIFOLDS.

LET $X$ BE A DIFFERENTIABLE MANIFOLD AND $p \in X$. A TANGENT VECTOR TO $X$ AT $p$ IS A MAPPING

$$\nabla^p : C^\infty(X) \to \mathbb{R}$$

THAT SATISFIES

(a) **LINEARITY**: $\nabla^p(af + bg) = a\nabla^p(f) + b\nabla^p(g)$ $\forall a, b \in \mathbb{R}$ $\forall f, g \in C^\infty(X)$

(b) **LEIBNIZ PRODUCT RULE**: $\nabla^p(fg) = \nabla^p(f) \nabla^p(g) + \nabla^p(f) \nabla^p(g)$ $\forall f, g \in C^\infty(X)$.

THE SET OF ALL TANGENT VECTORS TO $X$ AT $p$ IS DENOTED $T_p(X)$

AND CALLED THE TANGENT SPACE TO $X$ AT $p$. $T_p(X)$ HAS AN OBVIOUS REAL VECTOR SPACE STRUCTURE DEFINED BY

$$\nabla^p + \nabla^q \cdot f = \nabla^p(f) + \nabla^q(f)$$

$$\alpha \cdot \nabla^p \cdot f = \alpha \cdot \nabla^p(f)$$

EXAMPLES OF TANGENT VECTORS TO $X$ AT $p$ :

1. **LET $I$ BE AN OPEN INTERVAL IN $\mathbb{R}$ (THOUGHT OF AS AN OPEN SUBMANIFOLD OF $\mathbb{R}$ AND WITH STANDARD COORDINATE FUNCTION $t$)**. A SMOOTH MAP $\alpha : I \to X$ IS CALLED A SMOOTH CURVE IN $X$. SUPPOSE $\alpha$ GOES THROUGH
p at \( t = t_0 \in I \), i.e.,

\[ \alpha(t_0) = p \]

(Note that I should really write \((\alpha \circ \text{id}^{-1})(t_0) = p\), but this seems kind of silly.)

We define the velocity vector of \( \alpha \) at \( p \), denoted \( \alpha'(t_0) \), by

\[ \alpha'(t_0) : C^\infty(X) \to \mathbb{R} \]

by

\[ \alpha'(t_0)(f) = \left. \frac{d}{dt} \right|_{t=t_0} (f \circ \alpha) \]

(Note again that \( f \circ \alpha \) here should really be \( \text{id} \circ f \circ (\alpha \circ \text{id}^{-1}) \). Thus, \( \alpha'(t_0) \) just differentiates \( f \in C^\infty(X) \) along \( \alpha \) and evaluates at \( p \).

\( \alpha'(t) \) is in \( T_p(X) \) because \( \frac{d}{dt} \) is linear and satisfies the product rule.

Note: It is not obvious, but it can be proved that every element of \( T_p(X) \) is \( \alpha'(t_0) \) for some smooth curve \( \alpha \) through \( p \) (in general, there will be many such curves). We will prove this shortly.

2. Let \((U, \varphi)\) be a chart for \( X \) with \( p \in U \) and coordinate functions \( x'_1, \ldots, x'_n \). For each \( i = 1, \ldots, n \) define the \( i^{th} \) coordinate velocity vector at \( p \) by

\[ \partial_i |_p = \frac{\partial}{\partial x_i} |_p : C^\infty(X) \to \mathbb{R} \]

\[ \partial_i |_p (f) = \left. \frac{\partial}{\partial x_i} \right|_p (f) = D_i (f \circ \varphi^{-1}) (\varphi(p)) = \text{ordinary } x'_i\text{-partial derivative of the coordinate expression } f \circ \varphi^{-1} \text{ for } f \text{ at } \varphi(p). \]
NOTICE THAT, BY DEFINITION OF PARTIAL DERIVATIVES,

$$\frac{\partial}{\partial x^i} \bigg|_p = \alpha'(t_0)$$

WHERE $x$ AND $t_0$ ARE AS FOLLOWS: IF $\psi(p) = (x_0', ... , x_{i-1}', x_i', ... , x_n')$, THEN

$$\alpha(t) = \psi^{-1}(x_0', ... , t, ... , x_n')$$

AND

$$t_0 = x_0'. $$

THUS, $\frac{\partial}{\partial x^i} \bigg|_p$ IS THE VELOCITY VECTOR TO THE $i^{th}$ COORDINATE CURVE OF $\psi$ THROUGH $p$.

ALTHOUGH IT IS A BIT MISLEADING, IT IS CUSTOMARY TO WRITE

$$\frac{\partial}{\partial x^i} \bigg|_p (p) = \frac{\partial f}{\partial x^i} (p).$$

**BASIS THEOREM:** LET $\nu_p \in T_p(X)$ AND $\psi: \mathcal{U} \to \mathbb{R}^n$

A CHART AT $p$ WITH $x^i = \psi^{-1} \circ \psi_0$, $i = 1, ..., n$. THEN

$$\left\{ \frac{\partial}{\partial x^i} \bigg|_p, ... , \frac{\partial}{\partial x^n} \bigg|_p \right\}$$

IS A BASIS FOR $T_p(X)$ AND

$$\nu_p = \sum_{i=1}^{n} \nu_p(x^i) \frac{\partial}{\partial x^i} \bigg|_p.$$ 

IN PARTICULAR,

$$\dim T_p(X) = \dim X.$$
THE PROOF OF THE BASIS THEOREM REQUIRES A LITTLE WORK, BUT THE RESULT IS CRUCIAL SO WE'LL DO IT.

TO SIMPLIFY THINGS A BIT WE'LL COMPOSE \( \mathcal{C} \) WITH A TRANSLATION OF \( \mathbb{R}^n \) THAT CARRIES \( \mathcal{C}(p) \) TO \( 0 = (0, \ldots, 0) \) AND ASSUME AT THE OUTSET THAT

\[ \mathcal{C}(p) = 0 \in \mathbb{R}^n. \]

ALSO, BY SHRINKING \( \mathcal{U} \) IF NECESSARY WE CAN ASSUME THAT \( \mathcal{C}(\mathcal{U}) \) IS SOME OPEN \( \varepsilon \)-BALL ABOUT \( 0 \) IN \( \mathbb{R}^n \).

\[ \mathcal{C}(\mathcal{U}) = \mathcal{U}_\varepsilon(0) \]

NOW WE NEED A FEW PRELIMINARIES.

1. **BUMP FUNCTIONS.** LET \( \mathcal{W} \) BE AN OPEN SET IN THE SMOOTH MANIFOLD \( X \) AND \( p \in \mathcal{W} \). THEN \( \exists \) SMOOTH, REAL-VALUED FUNCTION \( g \) ON \( X \) THAT IS 1 ON SOME OPEN NEIGHBORHOOD OF \( p \) IN \( \mathcal{W} \), 0 OUTSIDE OF \( \mathcal{W} \) AND \( 0 \leq g(x) \leq 1 \) FOR ALL \( x \in X \).
NOTE: THE PROOF OF THIS AMOUNTS TO FINDING ANALOGOUS "SMOOTH SEPARATION FUNCTIONS" ON IR^n (THESE CAN BE BUILT FROM EXPONENTIAL FUNCTIONS) AND MOVING THEM TO X VIA CHARTS. SINCE THIS IS BASICALLY REAL ANALYSIS, I DO NOT WANT TO SPEND TIME DOING IT HERE.

2. Exercise 75: Let \( n_\mathbf{p} \in T_\mathbf{p}(X) \). Suppose \( f_1, f_2 \in C^\infty(X) \) agree on some open neighborhood \( W \) of \( \mathbf{p} \) in \( X \). Show that \( n_\mathbf{p}(f_1) = n_\mathbf{p}(f_2) \).

Hints: First show that it is enough to prove that if \( f \in C^\infty(X) \) vanishes on \( W \), then \( n_\mathbf{p}(f) = 0 \). Select a bump function \( g \) for \( \mathbf{p} \) in \( W \) and consider \( fg \in C^\infty(X) \). This is the constant function 0 on \( X \). Show that \( n_\mathbf{p}(0) = 0 \) and then expand \( n_\mathbf{p}(fg) \).

3. Exercise 76: Show that if \( f \in C^\infty(X) \) is constant on some neighborhood of \( \mathbf{p} \), then \( n_\mathbf{p}(f) = 0 \).

Hint: If \( 1 \) denotes the constant function with value 1 on \( X \), then \( n_\mathbf{p}(1) = n_\mathbf{p}(1 \cdot 1) = \ldots \)
Now let $C^\infty(p)$ denote the set of all real-valued functions defined and smooth on some open neighborhood of $p$ in $X$ (different open neighborhoods for different functions, in general).

For $f,g \in C^\infty(p)$ with $f$ defined on $W_f$ and $g$ defined on $W_g$, we define $f+g$ and $fg$ in $C^\infty(p)$ on $W_f \cap W_g$.

Given $f \in C^\infty(p)$ defined on $W$, there exists an $\tilde{f} \in C^\infty(X)$ which agrees with $f$ on some neighborhood of $p$ in $W$.

Exercise 77: Prove this.

If $\tilde{f}$ and $\tilde{f}'$ are two such extensions of $f$, then they agree on some neighborhood of $p$ so

$$\nabla_p(\tilde{f}) = \nabla_p(\tilde{f}')$$

$\forall \nabla_p \in T_p(M)$.

The upshot of all this is that any tangent vector $\nabla_p$ at $p$ (linear, Leibnitzian function $C^\infty(X) \to \mathbb{R}$) gives rise to a linear, Leibnitzian function

$$C^\infty(p) \to \mathbb{R}.$$ 

Some texts define a tangent vector at $p$ to be an operator on $C^\infty(p)$. 
4. The next preliminary is from calculus (sort of a "first order finite Taylor expansion"):

Let \( g \) be a smooth, real-valued function on \( U_\varepsilon(0) \). We claim that \( g \) can be written in the form

\[
g(u', \ldots, u^n) = g(0, \ldots, 0) + \sum_{i=1}^{n} u^i g_i(u', \ldots, u^n)
\]

where \( u', \ldots, u^n \) are standard coordinates on \( \mathbb{R}^n \) and \( g_1, \ldots, g_n \) are smooth functions on \( U_\varepsilon(0) \).

Here's why this is true. For each \( u = (u', \ldots, u^n) \in U_\varepsilon(0) \) and each \( t \in [0,1], \quad tu = (tu', \ldots, tu^n) \) is in \( U_\varepsilon(0) \) and

\[
\frac{d}{dt} g(tu', \ldots, tu^n) = \sum_{i=1}^{n} D_i g(tu', \ldots, tu^n) u^i.
\]

Integrating from \( t = 0 \) to \( t = 1 \) gives

\[
g(u', \ldots, u^n) - g(0, \ldots, 0) = \int_{0}^{1} \sum_{i=1}^{n} D_i g(tu', \ldots, tu^n) u^i dt
\]

\[
= \sum_{i=1}^{n} u^i \left( \int_{0}^{1} D_i g(tu', \ldots, tu^n) dt \right)
\]

Smoothness follows from differentiation under the integral sign.
Now back to the proof of

\[ n_p = \sum_{i=1}^{n} n_p(x^i) \frac{\partial}{\partial x^i} \bigg|_p. \]

We will evaluate \( n_p(f) \) for an arbitrary \( f \in C^\infty(X) \) and show that it equals \( \sum_{i=1}^{n} n_p(x^i) \frac{\partial}{\partial x^i} \bigg|_p (f) \).

Let \( f_\xi = f \circ \xi^{-1} : U_\xi(0) \to \mathbb{R} \) be the coordinate expression for \( f \) relative to \((U, \xi)\). Use \#4 to write

\[ f_\xi(x^1, ..., x^n) = f_\xi(o, ..., o) + \sum_{i=1}^{n} x^i g_i(x^1, ..., x^n) \]

where \( g_1, ..., g_n \) are smooth.

\[ (f \circ \xi^{-1})(x^1, ..., x^n) = (f \circ \xi^{-1})(o, ..., o) + \sum_{i=1}^{n} x^i (g_i \circ \xi^{-1})(x^1, ..., x^n) \]

Call this \( f_i : U \to \mathbb{R} \)

\[ f(x^1, ..., x^n) = f_p + \sum_{i=1}^{n} x^i f_i(x^1, ..., x^n) \]

\[ f(\xi(x^1, ..., x^n)) = f_p + \sum_{i=1}^{n} x^i f_i(\xi(x^1, ..., x^n)) \]

\[ f(\xi(x^1, ..., x^n)) = f_p + \sum_{i=1}^{n} (\pi^i \circ \xi)(\xi(x^1, ..., x^n)) f_i(\xi(x^1, ..., x^n)) \]

Thus, on \( U \),

\[ f = f_p + \sum_{i=1}^{n} x^i f_i \]

where \( f_p \) is now regarded as a constant function on \( U \).
NOW, LET'S COMPUTE

\[ \frac{2}{\partial x^i} \bigg|_p (f) = \frac{2}{\partial x^i} \bigg|_p (f(p)) + \sum_{i=1}^{n} x^i f_i(p) \]

\[ = \frac{2}{\partial x^i} \bigg|_p (f(p)) + \frac{2}{\partial x^i} \bigg|_p \left( \sum_{i=1}^{n} x^i f_i(p) \right) \]

\[ = 0 + \sum_{i=1}^{n} \frac{2}{\partial x^i} \bigg|_p (x^i f_i) \]

\[ = \sum_{i=1}^{n} (x^i f_i) \left( \frac{2}{\partial x^i} \bigg|_p (f(p)) + f_i(p) \frac{2}{\partial x^i} \bigg|_p (x^i) \right) \]

\[ \uparrow \]

\[ \downarrow \]

\[ = f_i(p) \]

\[ = f_i(p) \]

AND

\[ \nabla_p (f) = \nabla_p (f(p)) + \sum_{i=1}^{n} x^i f^i \]

\[ = \nabla_p (f(p)) + \sum_{i=1}^{n} \nabla_p (x^i f^i) \]

\[ = 0 + \sum_{i=1}^{n} (x^i f_i) \nabla_p (f(p)) + f_i(p) \nabla_p (x^i) \]

\[ \uparrow \]

\[ \downarrow \]

\[ = \sum_{i=1}^{n} \nabla_p (x^i) f_i(p) \]

\[ = \sum_{i=1}^{n} \nabla_p (x^i) \frac{2}{\partial x^i} \bigg|_p (f) \]


\[ \text{SINCE } f \text{ WAS ARBITRARY,} \]

\[ \nabla_p = \sum_{i=1}^{n} \nabla_p (x^i) \frac{2}{\partial x^i} \bigg|_p \]

\[ \text{AS REQUIRED.} \]
We've shown that \( \{ \frac{2}{\partial x^i} |_p, \ldots, \frac{2}{\partial x^n} |_p \} \) spans \( T_p(X) \).

Exercise 79: Use \( \frac{3}{\partial x^i} |_p (x^i) = \delta^i | \) to show that \( \{ \frac{2}{\partial x^i} |_p, \ldots, \frac{2}{\partial x^n} |_p \} \) is linearly independent.

This completes the proof of the Basis Theorem.

Exercise 80: Let \( n_p \in T_p(X) \). Show that there exists a smooth curve \( \gamma \) in \( X \), defined on some interval about 0 in \( \mathbb{R} \), such that \( \gamma(0) = p \) and \( \gamma'(0) = n_p \).

Hint: Choose a chart \((U, \phi)\) at \( p \), write \( n_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i} |_p \) (\( a_i = n_p(x^i) \)) and consider \( \gamma^{-1}(x^1(p) + \tau a_1, \ldots, x^n(p) + \tau a_n) \).

A vector field on \( X \) is a map \( V \) that assigns to each \( p \in X \) a tangent vector \( V(p) = V_p \in T_p(X) \).

If \((U, \phi)\) is a chart with coordinate functions \( x^1, \ldots, x^n \) and \( p \in U \), then, by the Basis Theorem,
\[ \nabla(p) = \nabla_p = \nabla_p(x^i) \frac{\partial}{\partial x^i} \]

**NOTE:** From this point on we will use the "Einstein summation convention" : repeated index, one superscript and one subscript, is summed over all the values the index can assume (a superscript in the denominator counts as a subscript).

The real-valued functions

\[ \nabla^i : U \to \mathbb{R} \]

\[ \nabla^i(p) = \nabla_p(x^i) \]

\[ i = 1, \ldots, n \]

are the components of \( \nabla \) relative to \((U, \mathcal{X})\).

\( \nabla \) is **continuous, smooth, ...** if its components are continuous, smooth, ... for all charts in some atlas for \( X \).

\[ T^r_1(TX) = \text{set of all smooth vector fields on } X \]

**Algebraic structure:**
\[ v, w \in T_x^*(X), \ a \in \mathbb{R}, \ f \in \mathcal{C}^\infty(X) : \]

\[ (v + w)(p) = v(p) + w(p) \]

\[ a v \in T_x^*(X) : (a v)(p) = a v(p) \]

\[ f v \in T_x^*(X) : (f v)(p) = f(p) v(p) \]

**Examples:**

1. \((U, \mathcal{U})\) a chart on \(X\) with coordinate functions \(x_1, \ldots, x^n\)

   define vector fields
   
   \[ \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n \]

   on the open submanifold \(U\) of \(X\) by
   
   \[ \frac{\partial}{\partial x_i}(p) = \left. \frac{\partial}{\partial x_i} \right|_p. \]

   Clearly smooth. If \(v \in T_x^*(X)\), then
   
   \[ v|_U = v^i \frac{\partial}{\partial x_i}; \]

   where \(v^1, \ldots, v^n\) are smooth real-valued functions on \(U\).

2. Let \(X = \mathbb{R}^2\) with its standard differentiable structure
   and use \(x\) and \(y\) (rather than \(x^1\) and \(x^2\)) as the
   coordinate functions for the standard chart \((\mathbb{R}^2, \text{id})\).
To describe a vector field on $\mathbb{R}^2$ we need only specify its component functions relative to $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Here's one example:

$$\mathbf{V} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

At any $p = (x_0, y_0) \in \mathbb{R}^2$

$$V(p) = -y(p) \frac{\partial}{\partial x}(p) + x(p) \frac{\partial}{\partial y}(p)$$

$$= -y(x_0, y_0) \frac{\partial}{\partial x}(x_0, y_0) + x(x_0, y_0) \frac{\partial}{\partial y}(x_0, y_0)$$

Is an operator on smooth functions given by

$$V(p)(f) = V_p(f) = V_{(x_0, y_0)}(f)$$

$$= -y(x_0, y_0) \frac{\partial f}{\partial x}(x_0, y_0) + x(x_0, y_0) \frac{\partial f}{\partial y}(x_0, y_0)$$

E.g., if $p = (2,3)$ and $f(x, y) = e^{x^2 + y^2}$, then

$$V(p)(f) = -3(2x e^{x^2 + y^2}(2,3)) + 2(2y e^{x^2 + y^2}(2,3))$$

$$= -6xe^{x^2 + y^2}(2,3) + 4y e^{x^2 + y^2}(2,3)$$

$$= -12e^5 + 12e^5$$

$$= 0$$

To get a "picture" of this vector field $\mathbf{V}$ on $\mathbb{R}^2$ we recall (page 4) that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the velocity
VECTORS TO THE STANDARD COORDINATE CURVES (HORIZONTAL AND VERTICAL STRAIGHT LINES), E.G.,

\[
\frac{\partial}{\partial x} \bigg|_{(x_0, y_0)} = \omega'(x_0)
\]

WHERE

\[
\omega(t) = (t, y_0).
\]

PICTURE THESE AS \( \vec{i} \) AND \( \vec{j} \) LOCATED AT \((x_0, y_0)\).

\[
V_{(x_0, y_0)} \frac{\partial}{\partial x} \bigg|_{(x_0, y_0)} + y_0 \frac{\partial}{\partial y} \bigg|_{(x_0, y_0)}
\]

\[
\frac{\partial}{\partial x} \bigg|_{(x_0, y_0)}
\]

\[
(x_0, y_0)
\]

\[
(x_0, y_0)
\]

THUS, \( V_{(x_0, y_0)} \) IS TANGENT TO THE CIRCLE ABOUT THE ORIGIN THROUGH \((x_0, y_0)\) (ORTHOGONAL TO THE "POSITION VECTOR" OF \((x_0, y_0)\)).

HERE'S ANOTHER WAY OF THINKING ABOUT VECTOR FIELDS:

LET \( V \in T'(TX) \). FOR EACH \( p \in X \),

\[
V_p : C^\infty(X) \rightarrow \mathbb{R}
\]

SO WE CAN THINK OF \( V \) AS A MAP

\[
V : C^\infty(X) \rightarrow C^\infty(X)
\]

\[
f \rightarrow Vf
\]
WHERE

\[(\mathcal{V}f)(p) = \mathcal{V}_p(f)\]

**Exercise 81**: Show that \(\mathcal{V}f\) is smooth \(\forall f \in C^\infty(X)\) and that

(a) \(\mathcal{V}(af + bg) = a\mathcal{V}f + b\mathcal{V}g\) \(\forall a, b \in \mathbb{R}\) \(\forall f, g \in C^\infty(X)\)

(b) \(\mathcal{V}(fg) = f\mathcal{V}g + (\mathcal{V}f)g\) \(\forall f, g \in C^\infty(X)\)

A **derivation** on \(C^\infty(X)\) is a map \(\Theta : C^\infty(X) \to C^\infty(X)\) that satisfies

(a) \(\Theta(af + bg) = a\Theta(f) + b\Theta(g)\) \(\forall a, b \in \mathbb{R}\) \(\forall f, g \in C^\infty(X)\)

(b) \(\Theta(fg) = f\Theta(g) + \Theta(f)g\) \(\forall f, g \in C^\infty(X)\)

Thus, \(\forall \in T^1(X)\) gives rise to a derivation of \(C^\infty(X)\).

**Exercise 82**: Show that, conversely, every derivation \(\Theta\) on \(C^\infty(X)\) arises in this way from some \(\forall \in T^1(X)\).

**Hint**: Define \(\forall\) at \(p \in X\) by \(\forall_p(f) = \Theta(f)(p)\).

Thus, a vector field on \(X\) is nothing other than a derivation on \(C^\infty(X)\). We give an important example of how this point of view can be used.
LET $V$ AND $W$ BE SMOOTH VECTOR FIELDS ON $X$ (THOUGHT OF AS DERIVATIONS ON $C^\infty(X)$).

$$V, W : C^\infty(X) \to C^\infty(X)$$

DEFINE THE LIE BRACKET OF $V$ AND $W$, DENOTED

$$[V, W] : C^\infty(X) \to C^\infty(X),$$

BY

$$[V, W]f = V(Wf) - W(Vf)$$

EXAMPLE: LET $(U, \phi)$ BE A CHART ON $X$ AND $\frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n}$ THE COORDINATE VECTOR FIELDS ON THE OPEN SUBMANIFOLD $U$.

THEN

$$[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} ]f = \frac{\partial}{\partial x^i} \left( \frac{\partial f}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left( \frac{\partial f}{\partial x^i} \right) = 0.$$

IF $[V, W] = 0$, THEN $V$ AND $W$ ARE SAID TO COMMUTE.

WE HAVE TO VERIFY THAT $[V, W]$ REALLY IS A DERIVATION/VECTOR FIELD. LINEARITY IS EASY:

$$[V, W](af + bg) = V(W(af + bg)) - W(V(af + bg))$$

$$= V(awf + bwg) - W(avf + bvg)$$

$$= av(Wf + Wg) - aw(Vf + Vg)$$

$$= a(V(Wf) + V(Wg)) + b(V(Wf) - W(Vg))$$

$$= a[V, W]f + b[V, W]g$$
Next,

\[ [v, w](fg) = v(w(fg)) - w(v(fg)) \]

\[ = v(fwg + (wf)g) - w(fvg + (vf)g) \]

\[ = v(fwg) + v((wf)g) - w(fvg) - w((vf)g) \]

\[ = f v(wg) + (vf)(wg) + (wf)(vg) + v(wf)g \]

\[ - f w(vg) - (wf)(vg) - (vf)(wg) - w(vf)g \]

\[ = f (v(wg) - w(vg)) + (v(wf) - w(vf))g \]

\[ = f [v, w]g + ([v, w]f)g \]

As required.

Exercise 83: Prove the following properties of the Lie bracket.

1. (\textbf{R-bilinearity}) For \( a, b \in \mathbb{R}, \ u, v, w \in \mathcal{P}(\mathcal{T}X) \)

\[ [av + bw, u] = a[v, u] + b[w, u] \]

\[ [v, aw + bu] = a[v, w] + b[v, u] \]

2. (\textbf{Skew-symmetry}) \[ [w, v] = -[v, w] \]

(So \([v, v] = 0 \) for any \( v \in \mathcal{P}(\mathcal{T}X) \))
3. **JACOBI IDENTIY**

\[ [u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0 \]

4. **FOR** \( f, g \in C^\infty(X) \),

\[ [f v, w] = f [v, w] - (w f) v \]

\[ [v, g w] = g [v, w] + (v g) w \]

5. **IN LOCAL COORDINATES**, IF \( v = v^j \frac{\partial}{\partial x^j} \) AND \( w = w^i \frac{\partial}{\partial x^i} \), THEN

\[ [v, w] = (v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}) \frac{\partial}{\partial x^i} \]

(DON'T FORGET THE SUMMATION CONVENTION!)

**REMARK:** A REAL VECTOR SPACE \( \mathcal{V} \) ON WHICH IS DEFINED AN OPERATION

\[ [\ , \ ] : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \]

(CALLED BRACKET) SATISFYING

1. **(R-BILINEARITY)**

\[ [a\sigma + b\nu, \mu] = a[\sigma, \mu] + b[\nu, \mu] \]

\[ [\sigma, a\mu + b\nu] = a[\sigma, \mu] + b[\sigma, \nu] \]

\( \forall a, b \in \mathbb{R} \ \forall \mu, \sigma, \nu \in \mathcal{V} \)

2. **(SKEW-SYMMETRY)**

\[ [\sigma, \nu] = -[\nu, \sigma] \ \forall \sigma, \nu \in \mathcal{V} \]

3. **JACOBI IDENTIY**

\[ [\mu, [\sigma, \nu]] + [\nu, [\mu, \sigma]] + [\sigma, [\mu, \nu]] = 0 \]

\( \forall \mu, \sigma, \nu \in \mathcal{V} \)

IS CALLED A **LIE ALGEBRA**.
Thus, $T(L_T)$ with the Lie bracket is a Lie algebra.

Exercise 84: Show that each of the following is a Lie algebra.

1. $\mathfrak{gl}(n, \mathbb{R}) = \text{all } n \times n \text{ real matrices with } [ , ] = \text{commutator} \quad ([AB] = AB - BA)$

2. $\mathfrak{so}(n) = \text{all } n \times n \text{ skew-symmetric matrices with } [ , ] = \text{commutator}$

3. $\mathbb{R}^3$ with $[ , ] = \text{cross product, i.e.,}$

   $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}$.

4. $\mathfrak{sl}(2) = \text{all } 2 \times 2 \text{ complex matrices that are skew-Hermitian}$
   
   $\left( \mathbf{A}^T = -\mathbf{A} \right) \text{ and trace free } \left( \text{tr}(\mathbf{A}) = 0 \right)$

Note: The entries are complex, but $\mathfrak{sl}(2)$ is a real vector space.