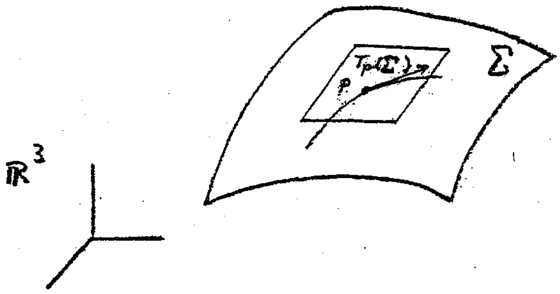
TANGENT VECTORS AND VECTOR FIELDS

A SMOOTH SURFACE Σ IN $\mathbb{R}^3$ HAS, AT EACH POINT $p \in \Sigma$, A <u>TANGENT PLANE</u> $T_p(\Sigma)$ CONSISTING OF ALL OF THE VELOCITY VECTORS TO ALL OF THE SMOOTH CURVES IN Σ THROUGH p.



THE VELOCITY VECTORS ARE THOUGHT OF AS ORDINARY VECTORS IN $\mathbb{R}^3$.

SIMILARLY, A DIFFERENTIABLE MANIFOLD X HAS, AT EACH POINT $p \in X$, A "TANGENT SPACE" $T_p(X)$, BUT THERE IS NO "AMBIENT EUCLIDEAN SPACE" LIKE $\mathbb{R}^3$ IN WHICH TO VIEW ITS ELEMENTS AS LIVING SO A DIFFERENT DEFINITION IS REQUIRED.

<u>MOTIVATION</u> : RETURN TO THE SMOOTH SURFACE Σ IN $\mathbb{R}^3$, SOME POINT $p \in \Sigma$ AND SOME TANGENT VECTOR $v_p \in T_p(\Sigma)$. ANY SMOOTH REAL-VALUED FUNCTION $f \in C^\infty(\Sigma)$ HAS A DERIVATIVE AT p WITH RESPECT TO v_p GIVEN BY

$$\nabla f(p) \cdot v_p$$

(IN CALCULUS, WHEN v_p IS A UNIT VECTOR THIS IS CALLED THE DIRECTIONAL DERIVATIVE OF f AT p IN THE DIRECTION v_p). THUS, v_p GIVES RISE TO A MAPPING FROM $C^\infty(\Sigma)$ TO $\mathbb{R}$ WHICH ASSIGNS TO EACH $f \in C^\infty(\Sigma)$ THIS NUMBER. USE THE SAME SYMBOL TO DENOTE THIS MAPPING :

$$v_p : C^\infty(\Sigma) \longrightarrow \mathbb{R}$$
$$v_p(f) = \nabla f(p) \cdot v_p \qquad \forall f \in C^\infty(\Sigma)$$

PROPERTIES :

(a) LINEARITY : $v_p(af + bg) = a\,v_p(f) + b\,v_p(g) \quad \forall a, b \in \mathbb{R} \ \forall f, g \in C^\infty(\Sigma)$

(b) LEIBNITZ PRODUCT RULE : $v_p(fg) = f(p)\,v_p(g) + v_p(f)\,g(p) \quad \forall f, g \in C^\infty(\Sigma)$.

IT IS NOT OBVIOUS, BUT ONE CAN PROVE THAT ANY MAPPING $C^\infty(\Sigma) \to \mathbb{R}$ SATISFYING THESE TWO PROPERTIES MUST BE $f \to \nabla f(p) \cdot V$ FOR A UNIQUE $V \in T_p(\Sigma)$.

THUS, WE CAN COMPLETELY IDENTIFY ELEMENTS OF $T_p(\Sigma)$ WITH MAPS $C^\infty(\Sigma) \to \mathbb{R}$ SATISFYING (a) AND (b). THIS THEN GIVES US A WAY OF DEFINING TANGENT VECTORS TO ARBITRARY MANIFOLDS.

LET X BE A DIFFERENTIABLE MANIFOLD AND $p \in X$. A <u>TANGENT</u> <u>VECTOR</u> <u>TO</u> X <u>AT</u> p IS A MAPPING

$$v_p : C^\infty(X) \to \mathbb{R}$$

THAT SATISFIES

(a) <u>LINEARITY</u> : $v_p(af+bg) = a\,v_p(f) + b\,v_p(g) \quad \forall a,b \in \mathbb{R}\ \forall f,g \in C^\infty(X)$

(b) <u>LEIBNITZ</u> <u>PRODUCT</u> <u>RULE</u> : $v_p(fg) = f(p)\,v_p(g) + v_p(f)\,g(p) \quad \forall f,g \in C^\infty(X)$.

THE SET OF ALL TANGENT VECTORS TO X AT p IS DENOTED

$$T_p(X)$$

AND CALLED THE <u>TANGENT</u> <u>SPACE</u> <u>TO</u> X <u>AT</u> p. $T_p(X)$ HAS AN OBVIOUS REAL VECTOR SPACE STRUCTURE DEFINED BY

$$(v_p + w_p)(f) = v_p(f) + w_p(f)$$

$$(a\,v_p)(f) = a\,(v_p(f)).$$

<u>EXAMPLES</u> <u>OF</u> <u>TANGENT</u> <u>VECTORS</u> <u>TO</u> X <u>AT</u> p :

1. LET I BE AN OPEN INTERVAL IN $\mathbb{R}$ (THOUGHT OF AS AN OPEN SUBMANIFOLD OF $\mathbb{R}$ AND WITH STANDARD COORDINATE FUNCTION t). A SMOOTH MAP $\alpha : I \to X$ IS CALLED A <u>SMOOTH</u> <u>CURVE</u> <u>IN</u> X. SUPPOSE α GOES THROUGH

p AT $t = t_0 \in I$, I.E.,

$$\alpha(t_0) = p$$

(NOTE THAT I SHOULD REALLY WRITE $(\alpha \circ id^{-1})(t_0) = p$, BUT THIS SEEMS KIND OF SILLY).

WE DEFINE THE <u>VELOCITY</u> <u>VECTOR</u> <u>OF</u> α <u>AT</u> p, DENOTED

$$\alpha'(t_0) : C^\infty(X) \longrightarrow \mathbb{R}$$

BY

$$\alpha'(t_0)(f) = \frac{d}{dt}(f \circ \alpha)\Big|_{t=t_0}$$

(NOTE AGAIN THAT $f \circ \alpha$ HERE SHOULD REALLY BE $id \circ f \circ \alpha \circ id^{-1}$). THUS, $\alpha'(t_0)$ JUST DIFFERENTIATES $f \in C^\infty(X)$ ALONG α AND EVALUATES AT p.

$\alpha'(t)$ IS IN $T_p(X)$ BECAUSE $\frac{d}{dt}$ IS LINEAR AND SATISFIES THE PRODUCT RULE.

> <u>NOTE</u>: IT IS NOT OBVIOUS, BUT IT CAN BE PROVED THAT <u>EVERY</u> ELEMENT OF $T_p(X)$ IS $\alpha'(t_0)$ FOR SOME SMOOTH CURVE α THROUGH p (IN GENERAL, THERE WILL BE MANY SUCH CURVES). WE WILL PROVE THIS SHORTLY.

2. LET (U, φ) BE A CHART FOR X WITH $p \in U$ AND COORDINATE FUNCTIONS $x^1, \ldots, x^n$. FOR EACH $i = 1, \ldots, n$ DEFINE THE i^{TH} <u>COORDINATE</u> <u>VELOCITY</u> <u>VECTOR</u> <u>AT</u> p BY

$$\partial_i|_p = \frac{\partial}{\partial x^i}\Big|_p : C^\infty(X) \longrightarrow \mathbb{R}$$

$$\partial_i|_p(f) = \frac{\partial}{\partial x^i}\Big|_p(f) = D_i(f \circ \varphi^{-1})(\varphi(p))$$

= ORDINARY x^i-PARTIAL DERIVATIVE OF THE COORDINATE EXPRESSION $f \circ \varphi^{-1}$ FOR f AT $\varphi(p)$.

NOTICE THAT, BY DEFINITION OF PARTIAL DERIVATIVES,

$$\frac{\partial}{\partial x^i}\Big|_p = \alpha'(t_0)$$

WHERE α AND t_0 ARE AS FOLLOWS: IF $\varphi(p) = (x_0^1, \dots, x_0^i, \dots, x_0^n)$, THEN

$$\alpha(t) = \varphi^{-1}(x_0^1, \dots, t, \dots, x_0^n)$$

AND

$$t_0 = x_0^i .$$

THUS, $\frac{\partial}{\partial x^i}\Big|_p$ IS THE VELOCITY VECTOR TO THE i^{TH} COORDINATE CURVE OF φ THROUGH p.

ALTHOUGH IT IS A BIT MISLEADING IT IS CUSTOMARY TO WRITE

$$\frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial f}{\partial x^i}(p).$$

<u>BASIS</u> <u>THEOREM</u>: LET $v_p \in T_p(X)$ AND $\varphi : U \to \mathbb{R}^n$ A CHART AT p WITH $x^i = \pi^i \circ \varphi$, $i = 1, \dots, n$. THEN $\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \}$ IS A BASIS FOR $T_p(X)$ AND

$$v_p = \sum_{i=1}^{n} v_p(x^i) \frac{\partial}{\partial x^i}\Big|_p .$$

IN PARTICULAR,

$$\dim T_p(X) = \dim X.$$

THE PROOF OF THE BASIS THEOREM REQUIRES A LITTLE WORK, BUT THE RESULT IS CRUCIAL SO WE'LL DO IT.

TO SIMPLIFY THINGS A BIT WE'LL COMPOSE φ WITH A TRANSLATION OF $\mathbb{R}^n$ THAT CARRIES $\varphi(p)$ TO $0 = (0, \dots, 0)$ AND ASSUME AT THE OUTSET THAT

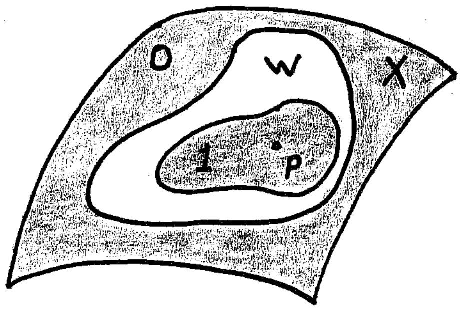
$$\varphi(p) = 0 \in \mathbb{R}^n.$$

ALSO, BY SHRINKING U IF NECESSARY WE CAN ASSUME THAT $\varphi(U)$ IS SOME OPEN ε-BALL ABOUT 0 IN $\mathbb{R}^n$.

$$\varphi(U) = U_\varepsilon(0)$$

NOW WE NEED A FEW PRELIMINARIES.

1. (BUMP FUNCTIONS) LET W BE AN OPEN SET IN THE SMOOTH MANIFOLD X AND $p \in W$. THEN $\exists$ SMOOTH, REAL-VALUED FUNCTION g ON X THAT IS 1 ON SOME OPEN NEIGHBORHOOD OF p IN W, 0 OUTSIDE OF W AND $0 \le g(x) \le 1$ FOR ALL $x \in X$.



NOTE : THE PROOF OF THIS AMOUNTS TO FINDING ANALOGOUS "SMOOTH SEPARATION FUNCTIONS" ON $\mathbb{R}^n$ (THESE CAN BE BUILT FROM EXPONENTIAL FUNCTIONS) AND MOVING THEM TO X VIA CHARTS. SINCE THIS IS BASICALLY REAL ANALYSIS, I DO NOT WANT TO SPEND TIME DOING IT HERE.

2. EXERCISE 75: LET $v_p \in T_p(X)$. SUPPOSE $f_1, f_2 \in C^\infty(X)$ AGREE ON SOME OPEN NEIGHBORHOOD W OF p IN X. SHOW THAT $v_p(f_1) = v_p(f_2)$.

HINTS : FIRST SHOW THAT IT IS ENOUGH TO PROVE THAT IF $f \in C^\infty(X)$ VANISHES ON W, THEN $v_p(f) = 0$. SELECT A BUMP FUNCTION g FOR p IN W AND CONSIDER $fg \in C^\infty(X)$. THIS IS THE CONSTANT FUNCTION 0 ON X. SHOW THAT $v_p(0) = 0$ AND THEN EXPAND $v_p(fg)$.

3. EXERCISE 76: SHOW THAT IF $f \in C^\infty(X)$ IS CONSTANT ON SOME NEIGHBORHOOD OF p, THEN $v_p(f) = 0$.

HINT : IF 1 DENOTES THE CONSTANT FUNCTION WITH VALUE 1 ON X, THEN $v_p(1) = v_p(1 \cdot 1) = \cdots$

NOW LET $C^\infty(p)$ DENOTE THE SET OF ALL REAL-VALUED FUNCTIONS DEFINED AND SMOOTH ON SOME OPEN NEIGHBORHOOD OF p IN X (DIFFERENT OPEN NEIGHBORHOODS FOR DIFFERENT FUNCTIONS, IN GENERAL).

FOR $f, g \in C^\infty(p)$ WITH f DEFINED ON W_f AND g DEFINED ON W_g, WE DEFINE $f+g$ AND fg IN $C^\infty(p)$ ON $W_f \cap W_g$.

GIVEN $f \in C^\infty(p)$ DEFINED ON W, THERE EXISTS AN $\tilde{f} \in C^\infty(X)$ WHICH AGREES WITH f ON SOME NEIGHBORHOOD OF p IN W

<u>EXERCISE</u> 77: PROVE THIS.

IF $\tilde{f}$ AND $\tilde{f}'$ ARE TWO SUCH EXTENSIONS OF f, THEN THEY AGREE ON SOME NEIGHBOORHOOD OF p SO

$$v_p(\tilde{f}) = v_p(\tilde{f}')$$

$\forall\, v_p \in T_p(M)$.

THE UPSHOT OF ALL THIS IS THAT ANY TANGENT VECTOR v_p AT p (LINEAR, LEIBNITZIAN FUNCTION $C^\infty(X) \rightarrow \mathbb{R}$) GIVES RISE TO A LINEAR, LEIBNITZIAN FUNCTION

$$C^\infty(p) \longrightarrow \mathbb{R}.$$

SOME TEXTS <u>DEFINE</u> A TANGENT VECTOR AT p TO BE AN OPERATOR ON $C^\infty(p)$.

4. THE NEXT PRELIMINARY IS FROM CALCULUS (SORT OF A "FIRST ORDER FINITE TAYLOR EXPANSION"):

LET g BE A SMOOTH, REAL-VALUED FUNCTION ON $U_\varepsilon(0)$. WE CLAIM THAT g CAN BE WRITTEN IN THE FORM

$$g(u^1,\dots,u^n) = g(0,\dots,0) + \sum_{i=1}^{n} u^i g_i(u^1,\dots,u^n)$$

WHERE $u^1,\dots,u^n$ ARE STANDARD COORDINATES ON $\mathbb{R}^n$ AND $g_1,\dots,g_n$ ARE SMOOTH FUNCTIONS ON $U_\varepsilon(0)$.

HERE'S WHY THIS IS TRUE. FOR EACH $u = (u^1,\dots,u^n) \in U_\varepsilon(0)$ AND EACH $t \in [0,1]$, $tu = (tu^1,\dots,tu^n)$ IS IN $U_\varepsilon(0)$ AND

$$\frac{d}{dt} g(tu^1,\dots,tu^n) = \sum_{i=1}^{n} D_i\, g(tu^1,\dots,tu^n)u^i .$$

INTEGRATING FROM $t=0$ TO $t=1$ GIVES

$$g(u^1,\dots,u^n) - g(0,\dots,0) = \int_0^1 \sum_{i=1}^{n} D_i g(tu^1,\dots,tu^n)u^i\, dt$$

$$= \sum_{i=1}^{n} u^i \underbrace{\left(\int_0^1 D_i g(tu^1,\dots,tu^n)\, dt \right)}_{g_i(u^1,\dots,u^n)}$$

SMOOTHNESS FOLLOWS FROM DIFFERENTIATION UNDER THE INTEGRAL SIGN

NOW BACK TO THE PROOF OF

$$v_p = \sum_{i=1}^{n} v_p(x^i) \frac{\partial}{\partial x^i}\Big|_p .$$

WE WILL EVALUATE $v_p(f)$ FOR AN ARBITRARY $f \in C^\infty(X)$ AND SHOW THAT IT EQUALS $\sum_{i=1}^{n} v_p(x^i) \frac{\partial}{\partial x^i}\Big|_p (f)$.

LET $f_\varphi = f \circ \varphi^{-1} : U_\varepsilon(0) \to \mathbb{R}$ BE THE COORDINATE EXPRESSION FOR f RELATIVE TO (U, φ). USE # 4 TO WRITE

$$f_\varphi(u^1, \ldots, u^n) = f_\varphi(0, \ldots, 0) + \sum_{i=1}^{n} u^i g_i(u^1, \ldots, u^n)$$

WHERE $g_1, \ldots, g_n$ ARE SMOOTH.

$$(f \circ \varphi^{-1})(u^1, \ldots, u^n) = (f \circ \varphi^{-1})(0, \ldots, 0) + \sum_{i=1}^{n} u^i \big((\underbrace{g_i \circ \varphi}_{\substack{\text{CALL THIS} \\ f_i : U \to \mathbb{R}}}) \circ \varphi^{-1} \big)(u^1, \ldots, u^n)$$

$$f(\varphi^{-1}(u^1, \ldots, u^n)) = f(p) + \sum_{i=1}^{n} u^i f_i(\varphi^{-1}(u^1, \ldots, u^n))$$

$$f(\varphi^{-1}(u^1, \ldots, u^n)) = f(p) + \sum_{i=1}^{n} (\pi^i \circ \varphi)(\varphi^{-1}(u^1, \ldots, u^n)) f_i(\varphi^{-1}(u^1, \ldots, u^n))$$

THUS, ON U,

$$f = f(p) + \sum_{i=1}^{n} x^i f_i$$

WHERE $f(p)$ IS NOW REGARDED AS A CONSTANT FUNCTION ON U.

NOW, LET'S COMPUTE

$$\frac{\partial}{\partial x^j}\Big|_p (f) = \frac{\partial}{\partial x^j}\Big|_p \left(f(p) + \sum_{i=1}^n x^i f_i \right)$$

NOTE: WE HAVE "GONE LOCAL" ON THE RIGHT-HAND SIDE.

$$= \frac{\partial}{\partial x^j}\Big|_p (f(p)) + \frac{\partial}{\partial x^j}\Big|_p \left(\sum_{i=1}^n x^i f_i \right)$$

$$= 0 + \sum_{i=1}^n \frac{\partial}{\partial x^j}\Big|_p (x^i f_i)$$

$$= \sum_{i=1}^n \Big(\underbrace{x^i(p)}_{0} \frac{\partial}{\partial x^j}\Big|_p (f_i) + f_i(p) \underbrace{\frac{\partial}{\partial x^j}\Big|_p (x^i)}_{\delta^{ij}} \Big)$$

(EXERCISE 78: PROVE THIS.)

$$= f_j(p)$$

AND

$$v_p(f) = v_p\left(f(p) + \sum_{i=1}^n x^i f_i\right) = v_p(f(p)) + \sum_{i=1}^n v_p(x^i f_i)$$

$$= 0 + \sum_{i=1}^n \big(\underbrace{x^i(p)}_{0}\, v_p(f_i) + f_i(p)\, v_p(x^i) \big)$$

$$= \sum_{i=1}^n v_p(x^i) f_i(p)$$

$$= \sum_{i=1}^n v_p(x^i) \frac{\partial}{\partial x^i}\Big|_p (f)$$

SINCE f WAS ARBITRARY,

$$v_p = \sum_{i=1}^n v_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$$

AS REQUIRED.

WE'VE SHOWN THAT $\{ \frac{\partial}{\partial x^1}\big|_p , \dots , \frac{\partial}{\partial x^n}\big|_p \}$ SPANS $T_p(X)$

<u>EXERCISE</u> 79: USE $\frac{\partial}{\partial x^i}\big|_p (x^j) = \delta^{ij}$ TO SHOW THAT $\{ \frac{\partial}{\partial x^1}\big|_p , \dots , \frac{\partial}{\partial x^n}\big|_p \}$ IS LINEARLY INDEPENDENT.

THIS COMPLETES THE PROOF OF THE BASIS THEOREM. $\square$

<u>EXERCISE</u> 80: LET $v_p \in T_p(X)$. SHOW THAT THERE EXISTS A SMOOTH CURVE α IN X, DEFINED ON SOME INTERVAL ABOUT 0 IN $\mathbb{R}$, SUCH THAT $\alpha(0) = p$ AND

$$\alpha'(0) = v_p \; .$$

<u>HINT</u>: CHOOSE A CHART (U, φ) AT p, WRITE $v_p = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\big|_p$ $(a^i = v_p(x^i))$ AND CONSIDER $\varphi^{-1}(x^1(p) + t a^1, \dots, x^n(p) + t a^n)$.

A <u>VECTOR FIELD</u> ON X IS A MAP V THAT ASSIGNS TO EACH $p \in X$ A TANGENT VECTOR

$$V(p) = V_p \in T_p(X) .$$

IF (U, φ) IS A CHART WITH COORDINATE FUNCTIONS $x^1, \dots, x^n$ AND $p \in U$, THEN, BY THE BASIS THEOREM,

$$V(p) = V_p = V_p(x^i) \frac{\partial}{\partial x^i}\Big|_p$$

<u>NOTE</u>: FROM THIS POINT ON WE WILL USE THE "EINSTEIN SUMMATION CONVENTION": REPEATED INDEX, ONE SUPERSCRIPT AND ONE SUBSCRIPT, IS SUMMED OVER ALL THE VALUES THE INDEX CAN ASSUME (A SUPERSCRIPT IN THE DENOMINATOR COUNTS AS A SUBSCRIPT).

THE REAL-VALUED FUNCTIONS

$$V^i : U \to \mathbb{R}$$

$$V^i(p) = V_p(x^i)$$

$$i = 1, \ldots, n$$

ARE THE <u>COMPONENTS</u> OF V RELATIVE TO (U, φ).

V IS <u>CONTINUOUS</u>, <u>SMOOTH</u>, ... IF ITS COMPONENTS ARE CONTINUOUS, SMOOTH, ... FOR ALL CHARTS IN SOME ATLAS FOR X.

$\Gamma(TX)$ = SET OF ALL SMOOTH VECTOR FIELDS ON X

ALGEBRAIC STRUCTURE :

$V, W \in \mathcal{T}(TX)$, $\quad a \in \mathbb{R}$, $\quad f \in C^\infty(X)$:

$$V + W \in \mathcal{T}(TX) \quad : \quad (V+W)(p) = V(p) + W(p)$$
$$aV \in \mathcal{T}(TX) \quad : \quad (aV)(p) = a\,V(p)$$
$$fV \in \mathcal{T}(TX) \quad : \quad (fV)(p) = f(p)\,V(p)$$

EXAMPLES :

1. (U, φ) A CHART ON X WITH COORDINATE FUNCTIONS $x^1, \ldots, x^n$

   DEFINE VECTOR FIELDS

$$\frac{\partial}{\partial x^i} \qquad , \quad i = 1, \ldots, n$$

   ON THE OPEN SUBMANIFOLD U OF X BY

$$\frac{\partial}{\partial x^i}(p) = \left.\frac{\partial}{\partial x^i}\right|_p \quad .$$

   CLEARLY SMOOTH. IF $V \in \mathcal{T}(TX)$, THEN

$$V|U = V^i \frac{\partial}{\partial x^i}$$

   WHERE $V^1, \ldots, V^n$ ARE SMOOTH REAL-VALUED FUNCTIONS ON U.

2. LET $X = \mathbb{R}^2$ WITH ITS STANDARD DIFFERENTIABLE STRUCTURE AND USE x AND y (RATHER THAN x^1 AND x^2) AS THE COORDINATE FUNCTIONS FOR THE STANDARD CHART $(\mathbb{R}^2, id)$.

TO DESCRIBE A VECTOR FIELD ON $\mathbb{R}^2$ WE NEED ONLY SPECIFY ITS COMPONENT FUNCTIONS RELATIVE TO $\frac{\partial}{\partial x}$ AND $\frac{\partial}{\partial y}$. HERE'S ONE EXAMPLE :

$$V = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

AT ANY $p = (x_0, y_0) \in \mathbb{R}^2$

$$V(p) = -y(p) \frac{\partial}{\partial x}(p) + x(p) \frac{\partial}{\partial y}(p)$$

$$= -y_0 \frac{\partial}{\partial x}\Big|_{(x_0,y_0)} + x_0 \frac{\partial}{\partial y}\Big|_{(x_0,y_0)}$$

IS AN OPERATOR ON SMOOTH FUNCTIONS GIVEN BY

$$V(p)(f) = V_p(f) = V_{(x_0,y_0)}(f)$$

$$= -y_0 \frac{\partial f}{\partial x}(x_0,y_0) + x_0 \frac{\partial f}{\partial y}(x_0,y_0)$$

E.G., IF $p = (2,3)$ AND $f(x,y) = e^{x^2+y^2}$, THEN

$$V(p)(f) = -3\,(2x e^{x^2+y^2})(2,3) + 2\,(2y e^{x^2+y^2})(2,3)$$

$$= -6x e^{x^2+y^2}(2,3) + 4y e^{x^2+y^2}(2,3)$$

$$= -12 e^{13} + 12 e^{13}$$

$$= 0$$

TO GET A "PICTURE" OF THIS VECTOR FIELD V ON $\mathbb{R}^2$ WE RECALL (PAGE 4) THAT $\frac{\partial}{\partial x}$ AND $\frac{\partial}{\partial y}$ ARE THE VELOCITY

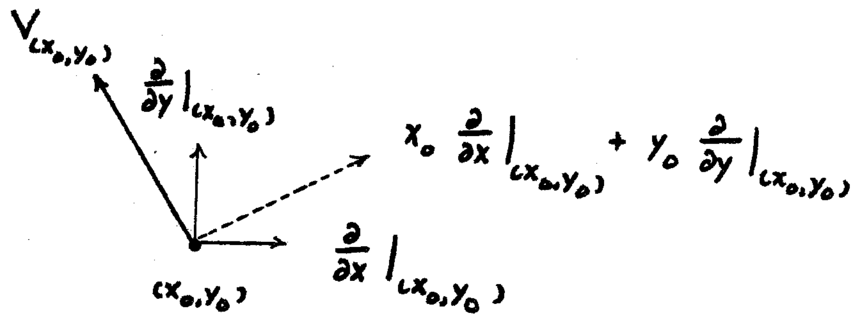
VECTORS TO THE STANDARD COORDINATE CURVES (HORIZONTAL AND VERTICAL STRAIGHT LINES), E.G.,

$$\frac{\partial}{\partial x}\Big|_{(x_0,y_0)} = \alpha'(x_0)$$

WHERE

$$\alpha(t) = (t, y_0).$$

PICTURE THESE AS $\vec{i}$ AND $\vec{j}$ LOCATED AT (x_0, y_0).



THUS, $V(x_0,y_0)$ IS TANGENT TO THE CIRCLE ABOUT THE ORIGIN THROUGH (x_0,y_0) (ORTHOGONAL TO THE "POSITION VECTOR" OF (x_0,y_0)).

HERE'S ANOTHER WAY OF THINKING ABOUT VECTOR FIELDS:

LET $V \in \Gamma(TX)$. FOR EACH $p \in X$,

$$V_p : C^\infty(X) \to \mathbb{R}$$

SO WE CAN THINK OF V AS A MAP

$$V : C^\infty(X) \to C^\infty(X)$$
$$f \longrightarrow Vf$$

WHERE

$$(Vf)(p) = V_p(f)$$

<u>EXERCISE</u> 81: SHOW THAT Vf IS SMOOTH $\forall f \in C^\infty(X)$ AND THAT

(a) $V(af+bg) = aVf + bVg \quad \forall a,b \in \mathbb{R} \ \forall f,g \in C^\infty(X)$

(b) $V(fg) = fVg + (Vf)g \quad \forall f,g \in C^\infty(X)$

A <u>DERIVATION</u> <u>ON</u> $C^\infty(X)$ IS A MAP

$$\mathcal{D} : C^\infty(X) \to C^\infty(X)$$

THAT SATISFIES

(a) $\mathcal{D}(af+bg) = a\mathcal{D}(f) + b\mathcal{D}(g) \quad \forall a,b \in \mathbb{R} \ \forall f,g \in C^\infty(X)$

(b) $\mathcal{D}(fg) = f\mathcal{D}(g) + \mathcal{D}(f)g \quad \forall f,g \in C^\infty(X)$

THUS, $V \in \Gamma(TX)$ GIVES RISE TO A DERIVATION OF $C^\infty(X)$.

<u>EXERCISE</u> 82: SHOW THAT, CONVERSELY, EVERY DERIVATION $\mathcal{D}$ ON $C^\infty(X)$ ARISES IN THIS WAY FROM SOME $V \in \Gamma(TX)$.
<u>HINT</u>: DEFINE V AT $p \in X$ BY $V_p(f) = \mathcal{D}(f)(p)$.

THUS, A VECTOR FIELD ON X IS NOTHING OTHER THAN A DERIVATION ON $C^\infty(X)$. WE GIVE AN IMPORTANT EXAMPLE OF HOW THIS POINT OF VIEW CAN BE USED.

LET V AND W BE SMOOTH VECTOR FIELDS ON X (THOUGHT OF AS DERIVATIONS ON $C^\infty(X)$).

$$V, W : C^\infty(X) \to C^\infty(X)$$

DEFINE THE <u>LIE BRACKET</u> OF V AND W, DENOTED

$$[V,W] : C^\infty(X) \to C^\infty(X),$$

BY

$$[V,W]f = V(Wf) - W(Vf)$$

<u>EXAMPLE</u>: LET (U, φ) BE A CHART ON X AND $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ THE COORDINATE VECTOR FIELDS ON THE OPEN SUBMANIFOLD U. THEN

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] f = \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) = 0.$$

IF $[V,W] = 0$, THEN V AND W ARE SAID TO <u>COMMUTE</u>.

WE HAVE TO VERIFY THAT $[V,W]$ REALLY IS A DERIVATION/VECTOR FIELD. LINEARITY IS EASY:

$$\begin{aligned}
[V,W](af+bg) &= V(W(af+bg)) - W(V(af+bg)) \\
&= V(aWf + bWg) - W(aVf + bVg) \\
&= aV(Wf) + bV(Wg) - aW(Vf) - bW(Vg) \\
&= a(V(Wf) - W(Vf)) + b(V(Wg) - W(Vg)) \\
&= a[V,W]f + b[V,W]g
\end{aligned}$$

NEXT,

$$
\begin{aligned}
[V,W](fg) &= V(W(fg)) - W(V(fg)) \\
&= V(fWg + (Wf)g) - W(fVg + (Vf)g) \\
&= V(fWg) + V((Wf)g) - W(fVg) - W((Vf)g) \\
&= fV(Wg) + \cancel{(Vf)(Wg)} + \cancel{(Wf)(Vg)} + V(Wf)g \\
&\quad - fW(Vg) - \cancel{(Wf)(Vg)} - \cancel{(Vf)(Wg)} - W(Vf)g \\
&= f(V(Wg) - W(Vg)) + (V(Wf) - W(Vf))g \\
&= f[V,W]g + ([V,W]f)g
\end{aligned}
$$

AS REQUIRED.

<u>EXERCISE</u> 83 : PROVE THE FOLLOWING PROPERTIES OF THE LIE BRACKET.

1. (<u>$\mathbb{R}$</u> - <u>BILINEARITY</u>) FOR $a, b \in \mathbb{R}$, $U, V, W \in \Gamma(TX)$

$$[aV + bW, U] = a[V,\ \ ] + b[W, U]$$

$$[V, aW + bU] = a[V,W] + b[V,U]$$

2. (<u>SKEW</u> - <u>SYMMETRY</u>) $\quad [W,V] = -[V,W]$

(SO $[V,V] = 0$ FOR ANY $V \in \Gamma(TX)$)

3. (JACOBI IDENTITY)

$$[U,[V,W]] + [W,[U,V]] + [V,[W,U]] = 0$$

4. FOR $f, g \in C^\infty(X)$,

$$[fV, W] = f[V,W] - (Wf)V$$

$$[V, gW] = g[V,W] + (Vg)W$$

5. IN LOCAL COORDINATES, IF $V = V^i \frac{\partial}{\partial x^i}$ AND $W = W^i \frac{\partial}{\partial x^i}$, THEN

$$[V,W] = \left(V^j \frac{\partial W^i}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

(DON'T FORGET THE SUMMATION CONVENTION !)

REMARK : A REAL VECTOR SPACE $\mathcal{V}$ ON WHICH IS DEFINED AN OPERATION

$$[\ ,\ ] : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$$

(CALLED BRACKET) SATISFYING

1. ($\mathbb{R}$-BILINEARITY) $[av + bw, u] = a[v,u] + b[w,u]$

   $[v, aw + bu] = a[v,w] + b[v,u]$

   $\forall\, a,b \in \mathbb{R}\ \forall\, u,v,w \in \mathcal{V}$

2. (SKEW-SYMMETRY) $[w, v] = -[v, w] \quad \forall\, v, w \in \mathcal{V}$

3. (JACOBI IDENTITY) $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$

   $\forall\, u, v, w \in \mathcal{V}$

IS CALLED A LIE ALGEBRA.

THUS, $T^1(TX)$ WITH THE LIE BRACKET IS A LIE ALGEBRA.

EXERCISE 84 : SHOW THAT EACH OF THE FOLLOWING IS A LIE ALGEBRA.

1. $\mathfrak{gl}(n, \mathbb{R})$ = ALL $n \times n$ REAL MATRICES WITH $[\ ,\ ]$ = COMMUTATOR ($[A,B] = AB - BA$)

2. $\mathfrak{so}(n)$: ALL $n \times n$ SKEW-SYMMETRIC MATRICES WITH $[\ ,\ ]$ = COMMUTATOR

3. $\mathbb{R}^3$ WITH $[\ ,\ ]$ = CROSS PRODUCT, I.E., $[\vec{x}, \vec{y}] = \vec{x} \times \vec{y}$.

4. $\mathfrak{su}(2)$ = ALL 2×2 COMPLEX MATRICES THAT ARE SKEW-HERMITIAN ($\bar{A}^T = -A$) AND TRACE FREE ($tr(A) = 0$)

NOTE : THE ENTRIES ARE COMPLEX, BUT $\mathfrak{su}(2)$ IS A REAL VECTOR SPACE.