

TECHNIQUES FOR COMPUTING LIMITS

EXAMPLES :

$$1. \lim_{x \rightarrow 3} x^2 = 3^2 = 9$$

$$2. \lim_{x \rightarrow -1} (x^2 + 2x) = (-1)^2 + 2(-1) = -1$$

POLYNOMIALS : $P(x) = a_n x^n + \dots + a_1 x + a_0$

$$\begin{aligned} \lim_{x \rightarrow a} P(x) &= P(a) \\ &= a_n a^n + \dots + a_1 a + a_0 \end{aligned}$$

$$3. \lim_{x \rightarrow 3} \frac{x^2 - 2x}{x+1} \rightarrow \frac{3^2 - 2(3)}{3+1} = \frac{3}{4}$$

$$4. \lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} \rightarrow \frac{1^2 - 4(1) + 3}{1^2 - 2(1) - 3} = \frac{0}{-4} = 0$$

$$5. \lim_{x \rightarrow 1} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} \rightarrow \frac{-4}{0}$$

SO THE LIMIT DOES NOT EXIST

RATIONAL FUNCTIONS: $P(x)$ AND $Q(x)$ POLYNOMIALS.

$$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \quad \text{UNLESS } Q(a) = 0$$

IF $Q(a) = 0$, BUT $P(a) \neq 0$, THE LIMIT DOES NOT EXIST.

IF $Q(a) = 0$ AND $P(a) = 0$ THE LIMIT IS SAID TO BE INDETERMINATE OF TYPE

$$\frac{0}{0}$$

AND MORE WORK IS REQUIRED TO DETERMINE THE LIMIT

E.G., ALL OF THE FOLLOWING ARE INDETERMINATE

$$\lim_{x \rightarrow 0} \frac{x^3}{x}, \quad \lim_{x \rightarrow 0} \frac{x}{x^3}, \quad \lim_{x \rightarrow 0} \frac{x^3}{x^3}, \quad \lim_{x \rightarrow 0} \frac{5x^3}{x^3}$$

6. $\lim_{x \rightarrow 1} \frac{2x+1}{x^2-1} \rightarrow \frac{2(1)+1}{1^2-1} = \frac{3}{0}$ SO THE LIMIT DNE.

NOTE THAT IN THIS CASE WE CAN DO BETTER:

$$\lim_{x \rightarrow 1^+} \frac{2x+1}{x^2-1} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{2x+1}{x^2-1} = -\infty$$

$$7. \quad \lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} \rightarrow 0$$

THIS TIME THE "EXTRA WORK" REQUIRED TO DETERMINE THE LIMIT IS PRETTY OBVIOUS :

FACTOR AND CANCEL

$$\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{(x-3)(x-1)}$$

$$= \lim_{x \rightarrow 3} \frac{x+1}{x-1}$$

NOTE : CANCELLING THE $x-3$ MEANS DIVIDING NUMERATOR AND DENOMINATOR BY $x-3$. THIS IS LEGITIMATE BECAUSE, IN

$$\lim_{x \rightarrow 3}, \quad x \neq 3.$$

$$= \frac{3+1}{3-1} = 2$$

$$8. \quad \lim_{t \rightarrow -2} \frac{t^3 + 8}{t + 2} \rightarrow 0$$

$$= \lim_{t \rightarrow -2} \frac{(t+2)(t^2 - 2t + 4)}{t+2}$$

$$= \lim_{t \rightarrow -2} (t^2 - 2t + 4)$$

$$= (-2)^2 - 2(-2) + 4 = 12$$

RECALL :

$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

$$9. \quad \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} \rightarrow 0$$

THIS TIME IT IS PROBABLY EASIEST TO

EXPAND

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} &= \lim_{h \rightarrow 0} \frac{(8 + 12h + 6h^2 + h^3) - 8}{h} \\ &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(12 + 6h + h^2)}{h} \\ &= \lim_{h \rightarrow 0} (12 + 6h + h^2) = 12 + 6 \cdot 0 + 0^2 = 12 \end{aligned}$$

$$10. \quad \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x} \rightarrow 0$$

THE STANDARD THING TO DO WITH A SQUARE ROOT IN A SUM OR DIFFERENCE IS

RATIONALIZE

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - 1}{x} \cdot \frac{\sqrt{1-x} + 1}{\sqrt{1-x} + 1} = \lim_{x \rightarrow 0} \frac{(\sqrt{1-x})^2 - 1^2}{x(\sqrt{1-x} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{1-x-1}{x(\sqrt{1-x} + 1)} = \lim_{x \rightarrow 0} \frac{-x}{x(\sqrt{1-x} + 1)} = \lim_{x \rightarrow 0} \frac{-1}{\sqrt{1-x} + 1} \\ &= \frac{-1}{\sqrt{1-0} + 1} = -\frac{1}{2} \end{aligned}$$

$$11. \quad \lim_{t \rightarrow 0} \frac{\frac{1}{2+t^2} - \frac{1}{2}}{t} \rightarrow 0$$

SIMPLIFY THE COMPLEX FRACTION

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\frac{1}{2+t^2} - \frac{1}{2}}{t} &= \lim_{t \rightarrow 0} \frac{\frac{1}{2+t^2} - \frac{1}{2}}{t} \cdot \frac{2(2+t^2)}{2(2+t^2)} \\ &= \lim_{t \rightarrow 0} \frac{2 - (2+t^2)}{2t(2+t^2)} = \lim_{t \rightarrow 0} \frac{-t^2}{2t(2+t^2)} \\ &= \lim_{t \rightarrow 0} \frac{-t}{2(2+t^2)} \begin{matrix} \rightarrow 0 \\ \rightarrow 4 \end{matrix} = \frac{0}{4} = 0 \end{aligned}$$

$$12. \quad f(x) = \begin{cases} \frac{1}{x+2} & , x < -2 \\ x^2 - 5 & , -2 < x \leq 3 \\ \sqrt{x+13} & , x > 3 \end{cases}$$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{1}{x+2} = -\infty$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x^2 - 5) = 0^2 - 5 = -5$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 - 5) = 3^2 - 5 = 4$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \sqrt{x+13} = \sqrt{3+13} = 4$$

THUS,
 $\lim_{x \rightarrow 3} f(x) = 4$