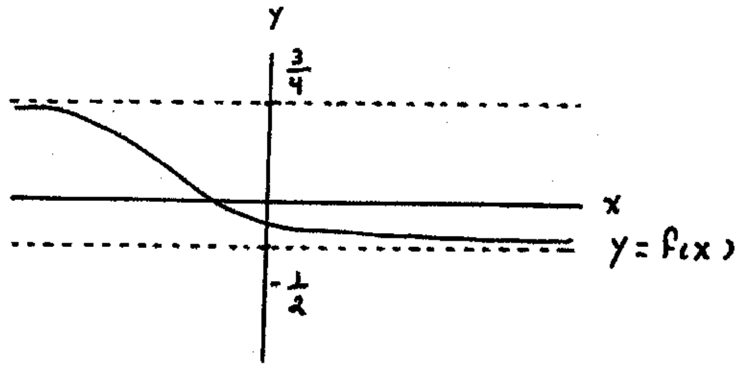
MORE TECHNIQUES FOR COMPUTING LIMITS

RECALL:



$$\lim_{x\to\infty} f(x) = -\frac{1}{2}$$

$$\lim_{x\to-\infty} f(x) = \frac{3}{4}$$

EXAMPLES:

1. $\lim_{x\to\infty} \frac{1}{x} = 0$

2. $\lim_{x\to-\infty} \frac{1}{x} = 0$

3. $\lim_{x\to\infty} \frac{2}{x^2} = 0$

4. $\lim_{x\to-\infty} \frac{2}{x^2} = 0$

IN GENERAL, IF k IS A CONSTANT AND $n > 0$, THEN

$$\lim_{x\to\pm\infty} \frac{k}{x^n} = 0$$

5. $\lim_{x\to\infty} \dfrac{2x+3}{5x-1} \begin{matrix}\longrightarrow \infty \\ \longrightarrow \infty\end{matrix}$

NOTE : LIMITS LIKE THIS ARE SAID TO BE INDETERMINATE OF TYPE

$$\frac{\infty}{\infty}$$

E.G., $\lim_{x\to\infty} \dfrac{x}{x^2}$, $\lim_{x\to\infty} \dfrac{x^2}{x^2}$, $\lim_{x\to\infty} \dfrac{5x^3}{x^3}$, ETC.

MORE WORK REQUIRED TO DETERMINE THE LIMIT.

$$\lim_{x\to\infty} \frac{2x+3}{5x-1} = \lim_{x\to\infty} \frac{2x+3}{5x-1} \, \frac{\frac{1}{x}}{\frac{1}{x}}$$

(DIVIDE TOP AND BOTTOM BY THE HIGHEST POWER OF X VISIBLE)

$$= \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5-\frac{1}{x}}$$

$$= \frac{2+0}{5-0} = \frac{2}{5}$$

6. $\lim_{x\to-\infty} \dfrac{2x+3}{5x-1} = \lim_{x\to-\infty} \dfrac{2+\frac{3}{x}}{5-\frac{1}{x}} = \dfrac{2+0}{5-0} = \dfrac{2}{5}$

7. $\lim_{x\to\infty} \dfrac{7x+2}{x^2-x} = \lim_{x\to\infty} \dfrac{7x+2}{x^2-x} \, \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$= \lim_{x\to\infty} \frac{\frac{7}{x}+\frac{2}{x^2}}{1-\frac{1}{x}} = \frac{0+0}{1-0} = \frac{0}{1} = 0$$

8. $\lim\limits_{x\to-\infty} \dfrac{5-3x^3}{x^2+1} = \lim\limits_{x\to-\infty} \dfrac{5-3x^3}{x^2+1}\ \dfrac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \lim\limits_{x\to-\infty} \dfrac{\frac{5}{x^3}-3}{\frac{1}{x}+\frac{1}{x^3}} \begin{array}{l}\longrightarrow -3\\ \longrightarrow 0\end{array}$ SO THE LIMIT DOES NOT EXIST

NOTE : WE CAN DO BETTER THIS TIME. FOR $x \to -\infty$, $\frac{5}{x^3}$, $\frac{1}{x}$ AND $\frac{1}{x^3}$ ARE ALL NEGATIVE (AND SO IS -3) SO

$$\lim_{x\to-\infty} \frac{5-3x^3}{x^2+1} = \infty$$

9. $\lim\limits_{x\to\infty} \sqrt{\dfrac{5x^2-3x}{2x^2+3x}} = \lim\limits_{x\to\infty} \sqrt{\dfrac{5x^2-3x}{2x^2+3x}\ \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}}$

$= \lim\limits_{x\to\infty} \sqrt{\dfrac{5-\frac{3}{x}}{2+\frac{3}{x}}} = \sqrt{\dfrac{5-0}{2+0}} = \sqrt{\dfrac{5}{2}}$

10. $\lim\limits_{x\to\infty} \dfrac{2-x}{\sqrt{7+6x^2}} = \lim\limits_{x\to\infty} \dfrac{2-x}{\sqrt{7+6x^2}}\ \dfrac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{2}{x}-1}{\sqrt{\frac{7}{x^2}+6}} = \dfrac{0-1}{\sqrt{0+6}} = -\dfrac{1}{\sqrt{6}}$

11. $\lim\limits_{x\to-\infty} \dfrac{2-x}{\sqrt{7+6x^2}} = \lim\limits_{x\to-\infty} \dfrac{2-x}{\sqrt{7+6x^2}}\ \dfrac{\frac{1}{x}}{-\frac{1}{\sqrt{x^2}}}$ NOTE THE MINUS SIGN !

$= \lim\limits_{x\to-\infty} \dfrac{\frac{2}{x}-1}{-\sqrt{\frac{7}{x^2}+6}} = \dfrac{0-1}{-\sqrt{0+6}} = \dfrac{1}{\sqrt{6}}$

12. $\lim\limits_{x\to\infty} (\sqrt{x^2+3x} - x)$

NOTE: BOTH TERMS APPROACH ∞ AS $x \to \infty$. LIMITS LIKE THIS ARE SAID TO BE INDETERMINATE OF TYPE

$$\infty - \infty$$

MORE WORK IS REQUIRED TO DETERMINE THE LIMIT.

$$\lim_{x\to\infty} (\sqrt{x^2+3x} - x) = \lim_{x\to\infty} \frac{\sqrt{x^2+3x} - x}{1} \qquad \text{NOW RATIONALIZE}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+3x} - x}{1} \cdot \frac{\sqrt{x^2+3x} + x}{\sqrt{x^2+3x} + x} = \lim_{x\to\infty} \frac{(\sqrt{x^2+3x})^2 - x^2}{\sqrt{x^2+3x} + x}$$

$$= \lim_{x\to\infty} \frac{x^2+3x-x^2}{\sqrt{x^2+3x} + x} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+3x} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{3}{\frac{\sqrt{x^2+3x}}{x} + 1} = \lim_{x\to\infty} \frac{3}{\frac{\sqrt{x^2+3x}}{\sqrt{x^2}} + 1} = \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{3}{x}} + 1}$$

$$= \frac{3}{\sqrt{1+0} + 1} = \frac{3}{2}$$

13. $\lim\limits_{x\to\infty} (\sqrt{x^2+3x} + x) = \infty$

NEXT, A FEW REMARKS ON POLYNOMIALS AS $x \to \pm\infty$

14. $\lim_{x \to \infty} x^2 = \infty$

15. $\lim_{x \to -\infty} x^2 = \infty$

SAME FOR ANY EVEN POWER

16. $\lim_{x \to \infty} x^3 = \infty$

17. $\lim_{x \to -\infty} x^3 = -\infty$

SAME FOR ANY ODD POWER

18. $\lim_{x \to \infty} -3x^6 = -\infty$

19. $\lim_{x \to -\infty} -5x^3 = \infty$

20. $\lim_{x \to \infty} (4x^4 + 2x) = \infty$

21. $\lim_{x \to -\infty} (4x^4 + 2x) = \lim_{x \to -\infty} 4x^4$ (HIGHEST POWER DOMINATES)

$= \infty$

IN GENERAL,

$$\lim_{x \to \pm\infty} (a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0) = \lim_{x \to \pm\infty} a_n x^n$$

AND THIS LIMIT WILL BE ∞ OR $-\infty$ DEPENDING ON n AND a_n

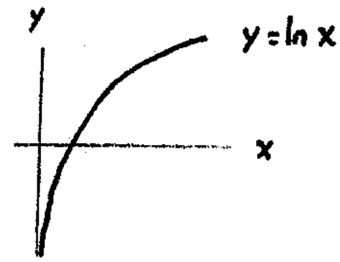
A FEW LIMITS THAT ARE WORTH REMEMBERING :

1. $\lim\limits_{x \to \pm\infty} \sin x$ AND $\lim\limits_{x \to \pm\infty} \cos x$ DO NOT EXIST

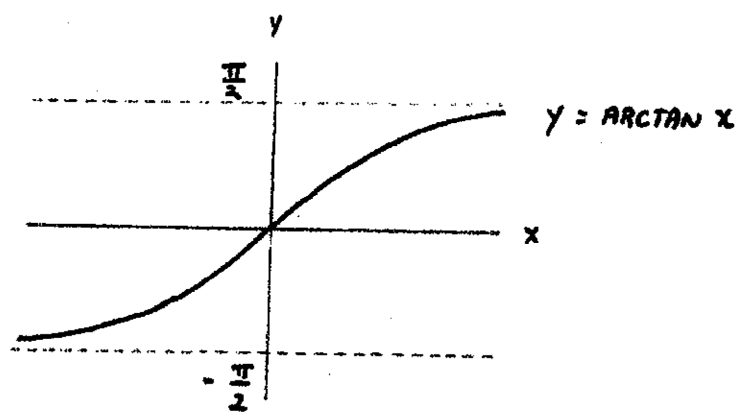
2. $\lim\limits_{x \to \infty} e^x = \infty$ AND $\lim\limits_{x \to \infty} \ln x = \infty$

3. $\lim\limits_{x \to -\infty} e^x = 0$ (NOTE THAT $\lim\limits_{x \to -\infty} \ln x$ MAKES NO SENSE)

4. $\lim\limits_{x \to 0^+} \ln x = -\infty$



5. $\lim\limits_{x \to \infty} \arctan x = \frac{\pi}{2}$ AND $\lim\limits_{x \to -\infty} \arctan x = -\frac{\pi}{2}$



<u>NOTE</u> : THE GRAPH OF $y = \arctan x$ COMES UP OFTEN ENOUGH THAT IT IS WORTH REMEMBERING.