

THE CHAIN RULE :

MOTIVATION :

$$1. \quad y = (x^2+1)^3 \\ = x^6 + 3x^4 + 3x^2 + 1$$

$$y' = 6x^5 + 12x^3 + 6x \\ = 6x(x^4 + 2x^2 + 1) \\ = 6x(x^2+1)^2 \\ = 3(x^2+1)^2(2x) \\ = 3(x^2+1)^2(x^2+1)'$$

⋮

$$\boxed{\begin{aligned} y &= (u(x))^n \\ y' &= n(u(x))^{n-1} u'(x) \end{aligned}}$$

$$2. \quad y = \sin 2x \\ = 2 \sin x \cos x$$

$$y' = 2(-\sin^2 x + \cos^2 x) \\ = 2(\cos^2 x - \sin^2 x) \\ = 2 \cos 2x \\ = \cos 2x (2x)'$$

⋮

$$\boxed{\begin{aligned} y &= \sin u(x) \\ y' &= \cos u(x) u'(x) \end{aligned}}$$

BOTH ARE EXAMPLES OF THE "CHAIN RULE"

MORE EXAMPLES :

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|----|-----------------|---------------|-----------------------------------|
| 1. | $y = \cos u(x)$ | \Rightarrow | $y' = -\sin u(x) u'(x)$ |
| 2. | $y = \tan u(x)$ | \Rightarrow | $y' = \sec^2 u(x) u'(x)$ |
| 3. | $y = \cot u(x)$ | \Rightarrow | $y' = -\csc^2 u(x) u'(x)$ |
| 4. | $y = \sec u(x)$ | \Rightarrow | $y' = \sec u(x) \tan u(x) u'(x)$ |
| 5. | $y = \csc u(x)$ | \Rightarrow | $y' = -\csc u(x) \cot u(x) u'(x)$ |

THE GENERAL CHAIN RULE :

$$y = f(u(x)) \Rightarrow y' = f'(u(x))u'(x)$$

WE'LL LOOK AT SOME EXAMPLES FIRST AND THEN I'LL TRY TO GIVE YOU A REASON TO BELIEVE THAT THIS IS TRUE.

1. $y = 4 \cos(x^3)$

$$\begin{aligned} y' &= 4 (\cos(x^3))' = 4 (-\sin(x^3) (x^3)') = 4 (-\sin(x^3)(3x^2)) \\ &= -12x^2 \sin(x^3) \end{aligned}$$

2. $y = (x^2 - x + 1)^{23}$

$$y' = 23(x^2 - x + 1)^{22} (x^2 - x + 1)' = 23(x^2 - x + 1)^{22} (2x - 1)$$

3. $s = \sqrt{1 + \tan t} = (1 + \tan t)^{\frac{1}{2}}$

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{2} (1 + \tan t)^{-\frac{1}{2}} (1 + \tan t)' = \frac{1}{2} (1 + \tan t)^{-\frac{1}{2}} \sec^2 t \\ &= \frac{\sec^2 t}{2\sqrt{1 + \tan t}} \end{aligned}$$

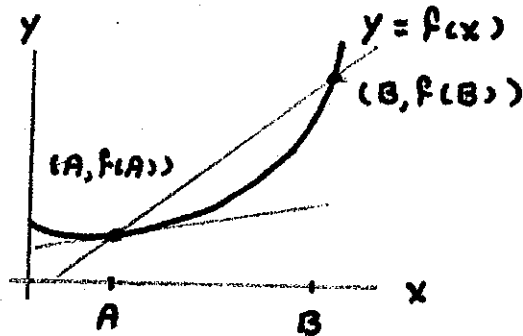
4. $R = \sec(\tan s)$

$$\begin{aligned} \frac{dR}{ds} &= \sec(\tan s) \tan(\tan s) (\tan s)' \\ &= \sec(\tan s) \tan(\tan s) \sec^2 s \end{aligned}$$

WE'LL DO SOME MORE COMPLICATED EXAMPLES SHORTLY, BUT FIRST WE SHOULD TRY TO UNDERSTAND WHY IT'S TRUE.

A CAREFUL PROOF IS SURPRIZINGLY TRICKY SO I'LL JUST TRY TO GIVE YOU A REASON TO BELIEVE.

NOTE :



$$f'(A) = \lim_{B \rightarrow A} \frac{f(B) - f(A)}{B - A}$$

NOW, ASSUMING f AND u ARE DIFFERENTIABLE,

$$\begin{aligned} (f(u(x)))' &= \lim_{h \rightarrow 0} \frac{f(u(x+h)) - f(u(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(u(x+h)) - f(u(x))}{u(x+h) - u(x)} \frac{u(x+h) - u(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(\overset{B}{u(x+h)}) - f(\overset{A}{u(x)})}{\underset{B}{u(x+h)} - \underset{A}{u(x)}} \frac{u(x+h) - u(x)}{h} \\ &= f'(u(x)) u'(x) \end{aligned}$$

BECAUSE u DIFFERENTIABLE $\Rightarrow u$ CONTINUOUS $\Rightarrow u(x+h) \rightarrow u(x)$ AS $h \rightarrow 0$. (CAN YOU SEE WHERE THIS "PROOF" MIGHT RUN INTO TROUBLE?)

SOME MORE EXAMPLES :

$$1. f(x) = \left(\frac{1+x^2}{1-x^2} \right)^{17}$$

$$\begin{aligned} f'(x) &= 17 \left(\frac{1+x^2}{1-x^2} \right)^{16} \left(\frac{1+x^2}{1-x^2} \right)' \\ &= 17 \left(\frac{1+x^2}{1-x^2} \right)^{16} \left(\frac{(1-x^2)(2x) - (1+x^2)(-2x)}{(1-x^2)^2} \right) \\ &= 17 \frac{(1+x^2)^{16}}{(1-x^2)^{16}} \left(\frac{4x}{(1-x^2)^2} \right) = \frac{68x(1+x^2)^{16}}{(1-x^2)^{18}} \end{aligned}$$

$$2. \text{ FIND } \frac{d^2y}{dx^2} \text{ IF } y = x \tan\left(\frac{1}{x}\right)$$

$$\begin{aligned} \frac{dy}{dx} &= (x \tan\left(\frac{1}{x}\right))' = x \left(\tan\left(\frac{1}{x}\right) \right)' + \tan\left(\frac{1}{x}\right) (x)' \\ &= x \sec^2\left(\frac{1}{x}\right) \left(\frac{1}{x}\right)' + \tan\left(\frac{1}{x}\right) \\ &= x \sec^2\left(\frac{1}{x}\right) \left(-\frac{1}{x^2}\right) + \tan\left(\frac{1}{x}\right) \\ &= -\left(\frac{1}{x}\right) \sec^2\left(\frac{1}{x}\right) + \tan\left(\frac{1}{x}\right) \end{aligned}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{1}{x} (\sec^2\left(\frac{1}{x}\right))' + \sec^2\left(\frac{1}{x}\right) \left(\frac{1}{x}\right)' \right] + \sec^2\left(\frac{1}{x}\right) \left(\frac{1}{x}\right)'$$

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$$\begin{aligned} &= -\frac{1}{x} (2 \sec\left(\frac{1}{x}\right) (\sec\left(\frac{1}{x}\right))') \\ &= -\frac{2}{x} \sec\left(\frac{1}{x}\right) \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) \left(\frac{1}{x}\right)' \\ &= -\frac{2}{x} \sec^2\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) \left(-\frac{1}{x^2}\right) \\ &= \frac{2 \sec^2\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right)}{x^3} \end{aligned}$$

3. FIND THE EQUATION OF THE TANGENT LINE TO THE GRAPH OF $y = f(x) = 3 \cot^3 x$ AT $x = \frac{\pi}{4}$

$$y - y_0 = m(x - x_0)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ f\left(\frac{\pi}{4}\right) & & \frac{\pi}{4} \\ 3 \cot^3 \frac{\pi}{4} & & \\ 3 & & \end{array}$$

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) : f'(x) &= 3(3 \cot^2 x (\cot x)') \\ &= 9 \cot^2 x (-\csc^2 x) \\ &= -9 \cot^2 x \csc^2 x \end{aligned}$$

$$f'\left(\frac{\pi}{4}\right) = -9(1)^2(\sqrt{2})^2 = -18$$

$$y - 3 = -18\left(x - \frac{\pi}{4}\right)$$

$$y = -18x + \frac{9\pi}{2} + 3$$

CHAIN RULE REPHRASED :

$$y = f(u) \text{ AND } u = u(x)$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}}$$

"LEIBNITZ NOTATION"