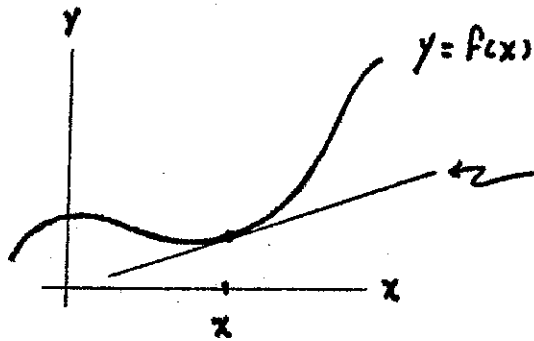


THE DERIVATIVE



SLOPE OF TANGENT LINE TO THE GRAPH OF f AT $(x, f(x))$

= INSTANTANEOUS RATE OF CHANGE OF f AT x

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (\text{PROVIDED THE LIMIT EXISTS})$$

NEW STUFF :

= THE DERIVATIVE OF f AT x

$$= f'(x)$$

$$= y'$$

$$= \frac{dy}{dx}$$

$$= \frac{d}{dx}(f(x))$$

EXAMPLE : LET $f(x) = \sqrt{x}$ FOR $x \geq 0$. THEN

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (\text{PROVIDED } x \neq 0) \end{aligned}$$

THUS,

$$f(x) = \sqrt{x} \text{ FOR } x \geq 0 \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \text{ FOR } x > 0$$

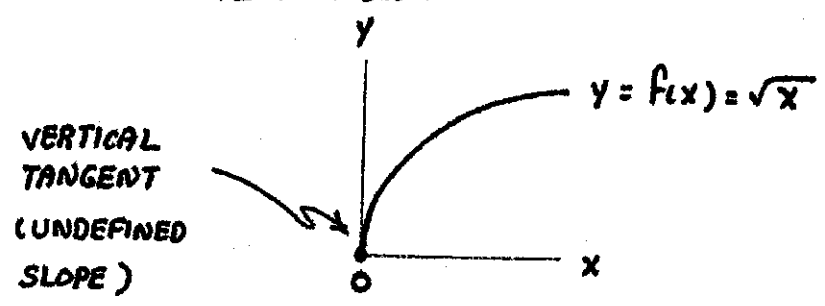
THE PROCESS OF FINDING DERIVATIVES OF FUNCTIONS IS CALLED
DIFFERENTIATION

IF A FUNCTION HAS A DERIVATIVE AT A POINT IT IS SAID TO BE
DIFFERENTIABLE

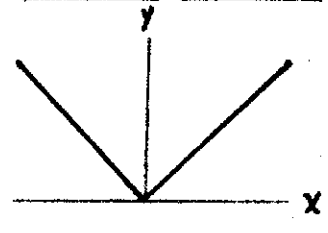
AT THAT POINT

E.G., $f(x) = \sqrt{x}$ IS DIFFERENTIABLE AT EVERY
POINT IN ITS DOMAIN EXCEPT $x=0$.

GEOMETRICAL REASON :



ANOTHER EXAMPLE : $f(x) = |x|$ IS NOT DIFFERENTIABLE AT $x=0$



$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h}$$

WHICH DOES NOT EXIST BECAUSE

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$$

ANOTHER WAY FOR DIFFERENTIABILITY TO FAIL :

THEOREM : IF $f(x)$ IS DIFFERENTIABLE AT x_0 , THEN $f(x)$ IS CONTINUOUS AT x_0 .

(SO, IF $f(x)$ IS NOT CONTINUOUS AT x_0 , THEN $f(x)$ IS NOT DIFFERENTIABLE AT x_0 .)

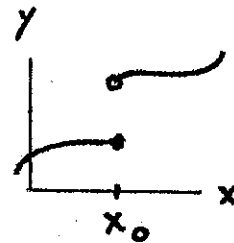
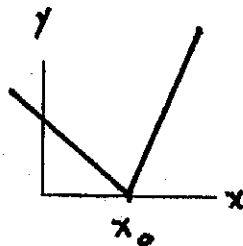
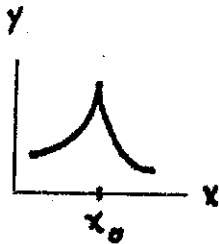
REASON : IF $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$, THEN

$$\begin{aligned} \lim_{x \rightarrow x_0} (f(x) - f(x_0)) &= \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) (x - x_0) \\ &= \left(\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \left(\lim_{x \rightarrow x_0} (x - x_0) \right) \\ &= f'(x_0) \cdot 0 \\ &= 0 \end{aligned}$$

SO

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

HERE, THEN, ARE THE WAYS IN WHICH $f(x)$ CAN FAIL TO BE DIFFERENTIABLE AT x_0 :



3. FIND THE POINT ON THE GRAPH OF $y = f(x) = x^2 + 2x - 1$
WHERE THE TANGENT LINE IS HORIZONTAL.

5.

TANGENT LINE HORIZONTAL \Leftrightarrow SLOPE OF TANGENT LINE ZERO

\Leftrightarrow DERIVATIVE ZERO

$$\Leftrightarrow f'(x) = 0$$

$$2x + 2 = 0 \quad (\text{FROM EXAMPLE 1})$$

$$x = -1$$

$$y = f(-1) = (-1)^2 + 2(-1) - 1$$

$$= -2$$

ANSWER : $(-1, -2)$

4. COMPUTE $g'(x)$ IF $g(x) = \frac{1}{x}$

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$