THE REAL NUMBER SYSTEM $\mathbb{R}$

NATURAL NUMBERS : $\mathbb{N} = \{1, 2, 3, \dots\}$

NOTE : $\mathbb{N}$ CAN BE CONSTRUCTED RIGOROUSLY FROM THE BASIC NOTIONS OF SET THEORY, BUT WE WILL NOT GET INTO THIS.

$\mathbb{N}$ IS CLOSED UNDER ADDITION

$$n, m \in \mathbb{N} \quad \Rightarrow \quad n+m \in \mathbb{N}$$

AND CLOSED UNDER MULTIPLICATION

$$n, m \in \mathbb{N} \quad \Rightarrow \quad nm \in \mathbb{N}.$$

$\mathbb{N}$ IS ORDERED BY $<$ (LESS THAN). THE RELATIONS $>$, $\leq$, AND $\geq$ ARE THEN DEFINED BY

$$m > n \iff n < m$$
$$n \leq m \iff n < m \text{ OR } n = m$$
$$m \geq n \iff n \leq m.$$

ONE OF THE MOST IMPORTANT PROPERTIES OF $\mathbb{N}$ IS

THE WELL-ORDERING PRINCIPLE : EVERY NONEMPTY SET OF NATURAL NUMBERS HAS A LEAST ELEMENT ; MORE PRECISELY, IF $S \subseteq \mathbb{N}$ AND $S \neq \emptyset$, THEN THERE IS AN $s_0 \in S$ SATISFYING $s_0 \leq s$ FOR EVERY $s \in S$.

E.G., IF $S = \{7n+3 : n \in \mathbb{N}\}$, THEN $s_0 = 10$.

THE WELL-ORDERING PRINCIPLE HAS THE FOLLOWING CONSEQUENCE, WHICH IS BEHIND ALL PROOFS BY INDUCTION.

THEOREM: LET n_0 BE SOME FIXED NATURAL NUMBER. SUPPOSE N IS A SUBSET OF $\mathbb{N}$ THAT SATISFIES

(a) $n_0 \in N$, AND, FOR $k \geq n_0$,

(b) $k \in N \Rightarrow k+1 \in N$.

THEN N CONTAINS ALL OF THE NATURAL NUMBERS GREATER THAN OR EQUAL TO n_0, I.E., $\{n_0, n_0+1, n_0+2, \dots\} \subseteq N$.

NOTE: INTUITIVELY, THE REASON IS

$n_0 \in N$ BY (a)

$n_0 \in N \Rightarrow n_0+1 \in N$ BY (b)

$n_0+1 \in N \Rightarrow n_0+2 \in N$ BY (b)

" ETC. "

PROOF: WE WILL PROVE THE THEOREM BY CONTRADICTION, I.E., WE WILL ASSUME THAT WHAT WE ARE TRYING TO PROVE IS FALSE AND DEDUCE FROM THIS ASSUMPTION A CONCLUSION THAT CANNOT BE TRUE.

THUS, LET US SUPPOSE THAT N SATISFIES (a) AND (b), BUT NEVERTHELESS DOES NOT CONTAIN ALL OF $\{n_0, n_0+1, n_0+2, \dots\}$. THEN THE SET

$$S = \{s \in \mathbb{N} : s \geq n_0 \text{ AND } s \notin N\}$$

IS NONEMPTY. BY THE WELL-ORDERING PRINCIPLE, S HAS A LEAST ELEMENT s_0.

BY (a), $s_0 \neq n_0$ SO $s_0 > n_0$ AND THEREFORE

$$s_0 - 1 \geq n_0 .$$

SINCE s_0 IS THE <u>LEAST</u> ELEMENT OF S AND $s_0 - 1 < s_0$, $s_0 - 1$ CANNOT BE IN S, BUT $s_0 - 1 \geq n_0$ SO THIS MEANS THAT $s_0 - 1$ MUST BE IN N. HOWEVER, (b) THEN IMPLIES THAT

$$(s_0 - 1) + 1 = s_0$$

IS ALSO IN N AND THIS CONTRADICTS THE FACT THAT $s_0 \in S$. THUS, OUR ASSUMPTION THAT $S \neq \emptyset$ MUST BE FALSE SO $\{n_0, n_0+1, n_0+2, \dots\} \subseteq N$. $\square$

A <u>PROOF</u> <u>BY</u> <u>INDUCTION</u> USES THIS THEOREM IN THE FOLLOWING WAY: LET $P(n)$ BE SOME STATEMENT ABOUT THE NATURAL NUMBER n. TO PROVE THAT $P(n)$ IS TRUE FOR ALL $n \geq n_0$ IT IS ENOUGH TO SHOW THAT

(a) $P(n_0)$ IS TRUE, AND, FOR $k \geq n_0$,

(b) $P(k)$ TRUE $\Rightarrow$ $P(k+1)$ TRUE.

<u>PROOF</u>: LET N BE THE SET OF THOSE NATURAL NUMBERS n FOR WHICH $P(n)$ IS TRUE. THEN $\{n_0, n_0+1, n_0+2, \dots\} \subseteq N$ IF (a) $n_0 \in N$, AND, FOR $k \geq n_0$, (b) $k \in N \Rightarrow k+1 \in N$. $\square$

EXAMPLES :

1. FOR EVERY NATURAL NUMBER $n \geq 1$,

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

(E.G., $1+2+3+4 = 10 = \frac{4(4+1)}{2}$).

THE STATEMENT IS TRUE FOR $n=1$ SINCE $1 = \frac{1(1+1)}{2}$. NOW ASSUME THAT THE STATEMENT IS TRUE FOR SOME $k \geq 1$, I.E., ASSUME THAT

$$1+2+\cdots+k = \frac{k(k+1)}{2}.$$

FROM THIS IT FOLLOWS THAT

$$\begin{aligned} 1+2+\cdots+k+(k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2}+1\right) = (k+1)\left(\frac{k+2}{2}\right) \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

AND THIS SIMPLY SAYS THAT THE STATEMENT MUST ALSO BE TRUE FOR $k+1$. IT FOLLOWS THAT THE STATEMENT MUST BE TRUE FOR ALL $n \geq 1$.

EXERCISES :

1. FIND A NONINDUCTIVE PROOF OF $1+2+\cdots+n = \frac{n(n+1)}{2}$.

   HINT: HERE, SO THE STORY GOES, IS HOW GAUSS DID IT FOR $n=500$ WHILE STILL IN ELEMENTARY SCHOOL.

$$1+2+3+\cdots+498+499+500$$

501

501

501

2. PROVE THAT FOR $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1).$$

3. PROVE THAT FOR $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2} \right)^2 .$$

2. FOR EVERY NATURAL NUMBER $n \geq 7$,

$$n! > 3^n.$$

(RECALL THAT $n!$ IS DEFINED BY $0! = 1$, $1! = 1$, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, $4! = 4 \cdot 3 \cdot 2 \cdot 1$, ..., $(n+1)! = (n+1)n!$, ...)

ONE CAN DO THE ARITHMETIC AND CHECK THAT $n! > 3^n$ IS <u>FALSE</u> FOR $n = 1, 2, 3, 4, 5, 6$, BUT

$$7! = 5040 > 2187 = 3^7.$$

SO IT IS TRUE FOR $n_0 = 7$.

NOW ASSUME THAT $k! > 3^k$. FROM THIS IT FOLLOWS THAT $(k+1)! > 3^{k+1}$ AS LONG AS $k > 2$. (AND, IN PARTICULAR, FOR $k \geq 7$). INDEED,

$$\begin{aligned} (k+1)! &= (k+1)k! \\ &> (k+1)3^k \\ &> 3 \cdot 3^k = 3^{k+1}. \end{aligned}$$

THUS, $n! > 3^n$ FOR $n \geq 7$.

EXERCISE 4 : FOR WHICH NATURAL NUMBERS n IS IT TRUE THAT

$$\frac{1}{n!} > \frac{8^n}{(2n)!}$$

(PROVE THAT YOUR ANSWER IS CORRECT) ?

3. ON A LARGE, FLAT FIELD, 2011 PEOPLE ARE POSITIONED SO THAT, FOR EACH PERSON, THE DISTANCES TO THE REMAINING PEOPLE ARE ALL DIFFERENT. EACH PERSON HAS A WATER PISTOL AND, WHEN A SIGNAL IS SOUNDED, EACH SHOOTS THE PERSON NEAREST TO HIM/HER. PROVE THAT AT LEAST ONE PERSON ON THE FIELD REMAINS DRY.

WE WILL SHOW THAT THE ONLY THING IMPORTANT ABOUT THE NUMBER OF PARTICIPANTS (2011) IS THAT IT IS ODD. MORE PRECISELY, WE PROVE BY INDUCTION ON $n = 1, 2, 3, \ldots$, THAT IF $2n+1$ PEOPLE PLAY THE GAME, THEN AT LEAST ONE REMAINS DRY.

FOR $n=1$ WE HAVE $2(1)+1 = 3$ PLAYERS. CALL THEM P_1, P_2 AND P_3. LET $d(P_1, P_2)$, $d(P_1, P_3)$ AND $d(P_2, P_3)$ BE THE DISTANCES BETWEEN THEM. THERE ARE ONLY THREE OF THEM SO WE CAN SELECT THE SMALLEST. BY RENUMBERING THE PEOPLE IF NECESSARY WE CAN ASSUME THE SMALLEST IS $d(P_2, P_3)$. THUS, P_2 AND P_3 SHOOT EACH OTHER. WHOEVER P_1 SHOOTS, NO ONE SHOOTS HIM SO HE REMAINS DRY.

NOW ASSUME THE RESULT FOR SOME $k \geq 1$, I.E., ASSUME THAT WHENEVER $2k+1$ PEOPLE PARTICIPATE AT LEAST ONE REMAINS DRY.

SUPPOSE $2(k+1)+1 = 2k+3$ PEOPLE PLAY THE GAME. DENOTE THEM $P_1, P_2, \ldots, P_{2k+1}, P_{2k+2}, P_{2k+3}$. LET THE DISTANCES BETWEEN THEM BE $d(P_i, P_j)$, $i, j = 1, \ldots, 2k+3$, $i \neq j$. THERE ARE ONLY FINITELY MANY SO THERE IS A SMALLEST WHICH, WITHOUT LOSS OF GENERALITY, WE CAN ASSUME IS $d(P_{2k+2}, P_{2k+3})$. THUS, P_{2k+2} AND P_{2k+3} SHOOT EACH OTHER.

NOW CONSIDER $P_1, P_2, \ldots, P_{2k+1}$. IF NONE OF THESE SHOOT EITHER P_{2k+2} OR P_{2k+3}, THEN $P_1, P_2, \ldots, P_{2k+1}$ ARE PLAYING THE GAME AMONG THEMSELVES SO OUR INDUCTION HYPOTHESIS IMPLIES THAT AT LEAST ONE OF THEM REMAINS DRY. IF ONE OF THEM SHOOTS EITHER P_{2k+2} OR P_{2k+3}, THEN THE $2k+1$ SHOTS FIRED BY $P_1, P_2, \ldots, P_{2k+1}$ CAN HIT AT MOST $2k$ OF THE PARTICIPANTS $P_1, P_2, \ldots, P_{2k+1}$ SO, AGAIN, AT LEAST ONE THEM REMAINS DRY AND THIS COMPLETES THE INDUCTION.

<u>OPTIONAL EXERCISE</u>: PROVE THAT THE SUM OF THE INTERIOR ANGLES IN A POLYGON WITH 1001 SIDES IS 999π.

THERE IS ANOTHER VERSION OF "PROOF BY INDUCTION" THAT IS SOMETIMES MORE CONVENIENT. IT IS PROVED FROM THE WELL-ORDERING PRINCIPLE IN EXACTLY THE SAME WAY SO I WILL LEAVE THIS FOR YOU TO WRITE OUT.

EXERCISE 4 : FOR WHICH NATURAL NUMBERS n IS IT TRUE THAT

$$\frac{1}{n!} > \frac{8^n}{(2n)!}$$

(PROVE THAT YOUR ANSWER IS CORRECT) ?

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NOW ASSUME THE RESULT FOR SOME $k \geq 1$, I.E., ASSUME THAT WHENEVER $2k+1$ PEOPLE PARTICIPATE AT LEAST ONE REMAINS DRY.

SUPPOSE $2(k+1)+1 = 2k+3$ PEOPLE PLAY THE GAME. DENOTE THEM $P_1, P_2, \ldots, P_{2k+1}, P_{2k+2}, P_{2k+3}$. LET THE DISTANCES BETWEEN THEM BE $d(P_i, P_j)$, $i, j = 1, \ldots, 2k+3$, $i \neq j$. THESE ARE ALL DIFFERENT SO THERE IS A SMALLEST WHICH, WITHOUT LOSS OF GENERALITY, WE CAN ASSUME IS $d(P_{2k+2}, P_{2k+3})$. THUS, P_{2k+2} AND P_{2k+3} SHOOT EACH OTHER.

NOW CONSIDER $P_1, P_2, \ldots, P_{2k+1}$. IF NONE OF THESE SHOOT EITHER P_{2k+2} OR P_{2k+3}, THEN $P_1, P_2, \ldots, P_{2k+1}$ ARE PLAYING THE GAME AMONG THEMSELVES SO OUR INDUCTION HYPOTHESIS IMPLIES THAT AT LEAST ONE OF THEM REMAINS DRY. IF ONE OF THEM SHOOTS EITHER P_{2k+2} OR P_{2k+3}, THEN THE $2k+1$ SHOTS FIRED BY $P_1, P_2, \ldots, P_{2k+1}$ CAN HIT AT MOST $2k$ OF THE PARTICIPANTS $P_1, P_2, \ldots, P_{2k+1}$ SO, AGAIN, AT LEAST ONE THEM REMAINS DRY AND THIS COMPLETES THE INDUCTION.

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<u>EXERCISE 5</u>: LET n_0 BE A FIXED NATURAL NUMBER AND SUPPOSE $P(n)$ IS SOME STATEMENT ABOUT THE NATURAL NUMBER n. PROVE THE FOLLOWING: TO SHOW THAT $P(n)$ IS TRUE FOR ALL $n \geq n_0$ IT IS ENOUGH TO SHOW THAT

(a) $P(n_0)$ IS TRUE AND, FOR $k \geq n_0$,

(b) $P(n_0), \ldots, P(k)$ ALL TRUE $\Rightarrow P(k+1)$ IS TRUE.

<u>EXAMPLE 4</u>: RECALL THAT A NATURAL NUMBER $p > 1$ IS <u>PRIME</u> IF IT CANNOT BE WRITTEN AS A PRODUCT OF TWO NATURAL NUMBERS, BOTH OF WHICH ARE LESS THAN p. HERE ARE A FEW PRIMES:

$$2, 3, 5, 7, 11, 13, 17, 19, \ldots, 73, \ldots, 97, \ldots, 2^{216091}-1, \ldots$$

(YOU'LL HAVE TO TRUST ME ON THIS LAST ONE). WE WILL PROVE THE FOLLOWING

<u>THEOREM</u>: EVERY NATURAL NUMBER $n \geq 2$ CAN BE WRITTEN AS A PRODUCT OF PRIMES (WHERE WE AGREE THAT THE "PRODUCT" HAS JUST ONE FACTOR IF n IS ITSELF PRIME).

<u>PROOF</u>: FOR $n = 2$ THIS IS CLEAR SINCE 2 IS ITSELF A PRIME.

<u>NOTE</u>: THE FIRST VERSION OF PROOF BY INDUCTION WOULD BE OF LITTLE USE NOW BECAUSE IT IS BASICALLY IMPOSSIBLE TO SAY ANYTHING ABOUT HOW THE PRIME FACTORS OF $k+1$ ARE RELATED TO THOSE OF k, E.G., $k = 2^{216091} - 1$ HAS JUST ONE, BUT $k+1$ HAS 216091.

NOW ASSUME THE RESULT IS TRUE OF ALL THE NATURAL NUMBERS $2, \ldots, k$ AND CONSIDER $k+1$. THERE ARE TWO POSSIBILITIES; EITHER $k+1$ IS PRIME, IN WHICH CASE WE ARE DONE, OR IT IS NOT. BUT IF $k+1$ IS NOT PRIME, THEN IT CAN BE WRITTEN AS $k+1 = ab$, WHERE BOTH a AND b ARE LESS THAN $k+1$ (AND OBVIOUSLY GREATER THAN 1). BY OUR INDUCTION HYPOTHESIS, BOTH a AND b CAN BE WRITTEN AS A PRODUCT OF PRIMES AND THEREFORE SO CAN $ab = k+1$. THUS, THE STATEMENT IS TRUE FOR ALL $n \geq 2$. □

> EXERCISE 6: PROVE THAT THERE ARE INFINITELY MANY PRIMES. HINT: ARGUE BY CONTRADICTION, ASSUMING $p_1, \ldots, p_k$ IS A COMPLETE LIST OF THE PRIMES AND THEN CONSIDER $p_1 \cdots p_k + 1$.

JUST FOR FUN: THE ANCIENT GREEKS WERE QUITE FOND OF NATURAL NUMBERS AND PARTICULARLY FOND OF THOSE LIKE 6 AND 28 THAT EQUAL THE SUM OF THEIR PROPER DIVISORS ($6 = 1+2+3$ AND $28 = 1+2+4+7+14$). THEY CALLED THESE PERFECT NUMBERS. NOTING THAT $6 = 2^1(2^2-1)$ AND $28 = 2^2(2^3-1)$, SEE IF YOU CAN FIND SOME MORE AND LEARN SOME GENERAL THINGS ABOUT THEM (AFTER 2000 YEARS OF TRYING NO ONE HAS YET BEEN ABLE TO DECIDE WHETHER OR NOT THERE ARE INFINITELY MANY OF THEM).

ADJOINING TO $\mathbb{N}$ AN ADDITIVE IDENTITY (0) AND AN ADDITIVE INVERSE $(-n)$ FOR EVERY $n \in \mathbb{N}$ ONE OBTAINS THE SET OF

INTEGERS : $\mathbb{Z} = \{ \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}$

$\mathbb{Z}$ IS CLOSED UNDER ADDITION AND MULTIPLICATION, BUT, OF COURSE, A QUOTIENT $\frac{m}{n}$ OF TWO INTEGERS (WITH $n \neq 0$) IS GENERALLY NOT AN INTEGER. THE LARGER SET OF NUMBERS OBTAINED BY FORMING ALL SUCH QUOTIENTS OF INTEGERS IS THE SET OF

RATIONAL NUMBERS : $\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$

AGAIN, $\mathbb{Q}$ IS CLOSED UNDER ADDITION AND MULTIPLICATION AND ORDERED BY $<$. WITH THESE OPERATIONS AND ORDER, $\mathbb{Q}$ HAS A GREAT MANY PROPERTIES THAT YOU ARE ALREADY VERY FAMILIAR WITH, BUT, FOR REASONS THAT MAY BECOME CLEAR ONLY A BIT LATER, WE WILL PICK OUT A FEW OF THEM TO LIST EXPLICITLY.

NOTE : IN THE JARGON OF ABSTRACT ALGEBRA THE PROPERTIES WE ARE ABOUT TO LIST ARE JUST THOSE THAT MAKE $\mathbb{Q}$ AN ORDERED FIELD. THE POINT IS THAT ONE CAN SHOW THAT ALL OF THE USUAL ARITHMETIC PROPERTIES OF THE RATIONAL NUMBERS CAN BE PROVED FROM THESE PROPERTIES ALONE.

(P1) ASSOCIATIVITY OF ADDITION :

$$a+(b+c) = (a+b)+c \qquad \forall a,b,c \in \mathbb{Q}$$

(P2) EXISTENCE OF AN ADDITIVE IDENTITY 0 :

$$a+0 = 0+a = a \qquad \forall a \in \mathbb{Q}$$

(P3) EXISTENCE OF ADDITIVE INVERSES :

$$a+(-a) = (-a)+a = 0 \qquad \forall a \in \mathbb{Q}$$

(P4) COMMUTATIVITY OF ADDITION :

$$a+b = b+a \qquad \forall a,b \in \mathbb{Q}$$

(P5) ASSOCIATIVITY OF MULTIPLICATION :

$$a(bc) = (ab)c \qquad \forall a,b,c \in \mathbb{Q}$$

(P6) EXISTENCE OF A MULTIPLICATIVE IDENTITY $1 \neq 0$:

$$a \cdot 1 = 1 \cdot a = a \qquad \forall a \in \mathbb{Q}$$

(P7) EXISTENCE OF MULTIPLICATIVE INVERSES :

$$aa^{-1} = a^{-1}a = 1 \qquad \forall a \in \mathbb{Q}, a \neq 0$$

(P8) COMMUTATIVITY OF MULTIPLICATION :

$$ab = ba \qquad \forall a,b \in \mathbb{Q}$$

(P9) DISTRIBUTIVITY :

$$(a+b)c = ac+bc \qquad \forall a,b,c \in \mathbb{Q}$$

(P10) TRICHOTOMY :

FOR ANY $a,b \in \mathbb{Q}$ ONE AND ONLY ONE OF THE FOLLOWING IS TRUE : $a=b$, $a<b$, $b<a$

(P11) TRANSITIVITY :

$$a<b \text{ AND } b<c \Rightarrow a<c \qquad \forall a,b,c \in \mathbb{Q}$$

(P12) $a<b \Rightarrow a+c < b+c \qquad \forall a,b,c \in \mathbb{Q}$

(P13) $a<b$ AND $c>0 \Rightarrow ac<bc \qquad \forall a,b,c \in \mathbb{Q}$

ALTHOUGH WE CERTAINLY DO NOT WANT TO SPEND ALL OF OUR TIME DERIVING "ALL OF THE USUAL ARITHMETIC PROPERTIES OF THE RATIONAL NUMBERS" FROM (P1) - (P13), WE WILL SPEND A FEW MINUTES PLAYING THIS GAME JUST TO SEE WHAT IT IS LIKE. THUS, WE WILL PRETEND, FOR A MOMENT, THAT WE KNOW NOTHING ABOUT $\mathbb{Q}$ EXCEPT (P1) - (P13) AND USE THESE TO PROVE A FEW THINGS THAT WE REALLY DO KNOW ALREADY.

1. THE ADDITIVE IDENTITY ELEMENT IS UNIQUE.

PROOF : SUPPOSE 0 AND $0'$ ARE TWO ELEMENTS OF $\mathbb{Q}$ THAT SATISFY $a+0=a$ AND $a+0'=a$ FOR EVERY $a \in \mathbb{Q}$. IN PARTICULAR, $0'+0=0'$ AND $0+0'=0$. BUT, BY (P4), $0'+0=0+0'$, SO $0'=0$. $\square$

EXERCISE 7 : PROVE THAT THE MULTIPLICATIVE IDENTITY ELEMENT IS UNIQUE.

2. FOR EVERY $a \in \mathbb{Q}$,

$$0 \cdot a = 0$$

PROOF :

$$
\begin{aligned}
0\cdot a &= (0+0)\cdot a && \text{(P2)}\\
0\cdot a &= 0\cdot a + 0\cdot a && \text{(P9)}\\
0\cdot a + (-0\cdot a) &= (0\cdot a + 0\cdot a) + (-0\cdot a) && \\
0 &= 0\cdot a + (0\cdot a + (-0\cdot a)) && \text{(P1)}\\
0 &= 0\cdot a + 0 && \text{(P3)}\\
0 &= 0\cdot a && \text{(P2)}
\end{aligned}
$$

$\square$

NOTE : IF YOU ARE WORRIED (AS YOU SHOULD BE) ABOUT WHAT JUSTIFIES THE STEP FROM LINE 2 TO LINE 3 ("ADDING THE SAME THING TO BOTH SIDES"), HERE IS THE EXPLANATION. ADDITION $+$ IS REALLY A FUNCTION THAT TAKES A PAIR (x,y) OF RATIONAL NUMBERS AND PRODUCES ANOTHER RATIONAL NUMBER $x+y$ (SUCH A FUNCTION IS CALLED A BINARY OPERATION ON $\mathbb{Q}$). CALL THE FUNCTION A FOR SECOND (SO $A(x,y) = x+y$). NOW SUPPOSE $x = x'$. THEN, FOR ANY y, (x,y) AND (x',y) ARE THE SAME PAIR SO THEY HAVE THE SAME VALUE UNDER A, I.E., $A(x,y) = A(x',y)$. THUS, $x = x'$ IMPLIES $x+y = x'+y$ FOR ANY y.

3. FOR ANY $a, b \in \mathbb{Q}$,

$$(-a)b = -(ab).$$

PROOF : FIRST NOTE THAT

$$\begin{aligned} (-a)b + ab &= ((-a)+a)b && (P9) \\ &= 0 \cdot b && (P3) \\ &= 0 && \#2 \text{ ABOVE} \end{aligned}$$

SO

$$\begin{aligned} ((-a)b + ab) + (-(ab)) &= 0 + (-(ab)) \\ (-a)b + (ab + (-(ab))) &= -(ab) && (P1) \\ (-a)b + 0 &= -(ab) && (P3) \\ (-a)b &= -(ab) && (P2) \end{aligned}$$

$\square$

EXERCISE 8 : PROVE THAT, FOR ANY $a, b \in \mathbb{Q}$,

$$(-a)(-b) = ab$$

4. FOR ANY $a \neq 0$ IN $\mathbb{Q}$, $a^2 > 0$.

PROOF : FIRST NOTE THAT IF $0 < a$, THEN (P13) AND #2 ABOVE IMPLY THAT $0 \cdot a < a \cdot a$, I.E., $0 < a^2$.

MOREOVER, IF $a < 0$, THEN (P12) GIVES $a + (-a) < 0 + (-a)$ SO $0 < -a$. BY WHAT WE HAVE JUST SHOWN AND EXERCISE 8,

$$0 < (-a)(-a) = aa = a^2.$$

THUS,

$$a \neq 0 \quad \Rightarrow \quad a^2 > 0. \qquad \square$$

EXERCISE 9 : PROVE THAT $1+1 \neq 0$.
(DON'T LAUGH. THE POINT HERE IS THAT THIS CANNOT BE PROVED FROM (P1)-(P9) ALONE, BUT REQUIRES THE ORDER AXIOMS.)

JUST TO SHOW YOU THAT THIS IS NOT AS SILLY AS IT MIGHT APPEAR :

EXERCISE 10 : LET $\mathbb{Z}_2 = \{[0], [1]\}$ BE A SET CONTAINING TWO ELEMENTS AND DEFINE ADDITION AND MULTIPLICATION ON THE SET BY THE FOLLOWING TABLES.

$+$	$[0]$	$[1]$
$[0]$	$[0]$	$[1]$
$[1]$	$[1]$	$[0]$

$\cdot$	$[0]$	$[1]$
$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$

SHOW THAT (P1)-(P9) ARE ALL SATISFIED, BUT $[1] + [1] = [0]$.

THIS SORT OF AXIOMATIC GAME COULD GO ON FOR QUITE SOME TIME, BUT IS NOT REALLY OUR BUSINESS HERE SO WE WILL NOW STOP PRETENDING THAT ALL WE KNOW ABOUT $\mathbb{Q}$ IS THE INFORMATION CONTAINED IN (P1) – (P13) AND GET BACK TO THE TASK AT HAND.

FROM OUR POINT OF VIEW THE MOST IMPORTANT PROPERTY OF $\mathbb{Q}$ IS THAT IT DOES NOT CONTAIN EVERYTHING WE WOULD LIKE, E.G.,

<u>THEOREM</u> (EUCLID): THERE IS NO RATIONAL NUMBER WHOSE SQUARE IS 2.

<u>PROOF</u>: WE ARGUE BY CONTRADICTION. SUPPOSE $\frac{m}{n} \in \mathbb{Q}$ AND $(\frac{m}{n})^2 = 2$. WE CAN ASSUME THAT ANY COMMON FACTORS OF m AND n HAVE BEEN CANCELLED, I.E., THAT $\frac{m}{n}$ IS REDUCED TO LOWEST TERMS. NOW,

$$\left(\frac{m}{n}\right)^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$$

SO m^2 IS EVEN AND THEREFORE m IS EVEN (THE SQUARE OF AN ODD INTEGER IS ODD BECAUSE $(2k+1)^2 = 2(2k^2+2k)+1$). THUS, WE CAN WRITE

$$m = 2k$$

FOR SOME INTEGER k. BUT THEN $m^2 = 2n^2 \Rightarrow (2k)^2 = 2n^2 \Rightarrow 4k^2 = 2n^2 \Rightarrow n^2 = 2k^2$ SO n^2 IS EVEN AND SO n IS EVEN, I.E.,

$$n = 2l$$

FOR SOME INTEGER l. BUT THEN m AND n BOTH HAVE A FACTOR OF 2 AND THIS CONTRADICTS THE FACT THAT $\frac{m}{n}$ IS REDUCED TO LOWEST TERMS.

$\square$

NOTE: THIS ARGUMENT CAN BE GENERALIZED TO SHOW THAT ANY NATURAL NUMBER THAT IS NOT THE SQUARE OF ANOTHER NATURAL NUMBER CAN ALSO NOT BE THE SQUARE OF ANY RATIONAL NUMBER.

IT IS TEMPTING TO PHRASE WHAT WE HAVE JUST PROVED BY SAYING THAT "$\sqrt{2}$ IS NOT RATIONAL". HOWEVER, THIS PRESUPPOSES THAT THERE IS SUCH A THING AS $\sqrt{2}$, THAT WE KNOW ALL ABOUT IT, AND THAT IT JUST DOESN'T HAPPEN TO LIVE IN $\mathbb{Q}$. BUT THEN WHERE DOES IT LIVE? EXACTLY WHAT KIND OF "THING" IS IT? OF COURSE, WE ALL THINK WE KNOW THE ANSWER: "$\sqrt{2}$ IS AN IRRATIONAL REAL NUMBER." BUT THIS BEGS THE QUESTION. WHAT IS A REAL NUMBER? CAN THESE BE CONSTRUCTED FROM $\mathbb{Q}$ THE WAY $\mathbb{Q}$ IS CONSTRUCTED FROM $\mathbb{Z}$, OR $\mathbb{Z}$ FROM $\mathbb{N}$?

IN FACT, SUCH A CONSTRUCTION IS POSSIBLE, ALTHOUGH IT IS BY NO MEANS AS SIMPLE. IT WAS FIRST CARRIED OUT BY RICHARD DEDEKIND IN THE LATE 19^{TH} CENTURY. EVERONE SHOULD GO THROUGH IT ONCE, BUT WE CANNOT AFFORD THE TIME TO DO IT HERE. FORTUNATELY, THE END RESULT OF DEDEKIND'S LABOR CAN BE RIGOROUSLY SUMMARIZED IN A VERY SIMPLE WAY. WE WILL SAY IT ONCE QUICKLY AND THEN ELABORATE.

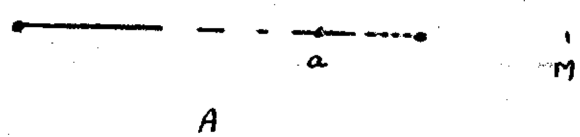
> THE SET $\mathbb{R}$ OF REAL NUMBERS IS A (IN FACT, THE ONLY) <u>COMPLETE</u>, <u>ORDERED</u> <u>FIELD</u>.

WHAT THIS MEANS IS THAT, LIKE $\mathbb{Q}$, THE ELEMENTS OF $\mathbb{R}$ CAN BE ADDED, MULTIPLIED, AND ORDERED IN SUCH A WAY THAT (P1) – (P13) ARE ALL SATISFIED, BUT, UNLIKE $\mathbb{Q}$, $\mathbb{R}$ SATISFIES ONE ADDITIONAL PROPERTY (P14), CALLED "COMPLETENESS". IT IS THIS PROPERTY THAT MAKES ANALYSIS POSSIBLE AND, IN PARTICULAR, IMPLIES THE EXISTENCE IN $\mathbb{R}$ OF SUCH THINGS AS $\sqrt{2}$.

WE NOW SET ABOUT DESCRIBING THIS "COMPLETENESS" PROPERTY, WHICH WILL BE OUR CONSTANT COMPANION FROM THIS POINT ON. WE WILL TAKE DEDEKIND'S WORD FOR THE FACT THAT $\mathbb{R}$ CAN BE CONSTRUCTED RIGOROUSLY AND WILL WORRY ONLY ABOUT SAYING WHAT "COMPLETENESS" MEANS AND THEN DERIVING ITS CONSEQUENCES.

A NONEMPTY SUBSET A OF $\mathbb{R}$ IS SAID TO BE <u>BOUNDED FROM ABOVE</u> IF THERE IS A REAL NUMBER M SUCH THAT

$$a \leq M \text{ FOR EVERY } a \in A$$



ANY SUCH NUMBER M IS CALLED AN <u>UPPER BOUND</u> FOR A.

E.G., IF $A = [0,1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ THEN ALL OF THE FOLLOWING ARE UPPER BOUNDS FOR A:

$$1,\ 2,\ 83,\ 10^{12}$$

AN UPPER BOUND M FOR A IS CALLED THE LEAST UPPER BOUND (OR SUPREMUM), OF A IF, WHENEVER M' IS AN UPPER BOUND FOR A, THEN $M \le M'$. IT IS DENOTED

$$\sup(A)$$

E.G., IF $A = [0,1)$, THEN $\sup(A) = 1$.

(P14) COMPLETENESS PROPERTY OF $\mathbb{R}$:

> EVERY NONEMPTY SUBSET OF $\mathbb{R}$ THAT IS BOUNDED FROM ABOVE HAS A LEAST UPPER BOUND.

WE WILL SEE A GREAT MANY APPLICATIONS OF COMPLETENESS BEFORE WE ARE THROUGH. WE BEGIN WITH ONE THAT MAY SEEM SCARCELY WORTHY OF PROOF, BUT IS SIGNIFICANT ENOUGH TO HAVE THE NAME OF ARCHIMEDES ATTACHED TO IT AND CANNOT BE PROVED FROM (P1) - (P13) ALONE.

THEOREM (THE ARCHIMEDIAN PRINCIPLE): THE SUBSET $\mathbb{N}$ OF $\mathbb{R}$ IS NOT BOUNDED FROM ABOVE.

PROOF: SUPPOSE TO THE CONTRARY THAT $\mathbb{N}$ IS BOUNDED FROM ABOVE. SINCE $\mathbb{N} \neq \emptyset$, COMPLETENESS IMPLIES THAT $\mathbb{N}$ MUST HAVE A LEAST UPPER BOUND M. THEN $n \le M$ FOR EVERY $n \in \mathbb{N}$. MOREOVER, $M-1 < M$ SO $M-1$ CANNOT BE AN UPPER BOUND FOR $\mathbb{N}$. THUS, THERE IS SOME $m \in \mathbb{N}$ WITH

SINCE $M > 1$ AND WE ARE ASSUMING $M^2 < 2$, $\frac{2-M^2}{2M+1}$ IS A POSITIVE REAL NUMBER. THUS, OUR COROLLARY ABOVE IMPLIES THAT WE CAN CHOOSE AN $n \in \mathbb{N}$ SUCH THAT

$$\frac{1}{n} < \frac{2-M^2}{2M+1}.$$

FOR SUCH AN n,

$$\begin{aligned}(M+\tfrac{1}{n})^2 &\le M^2 + \tfrac{1}{n}(2M+1) \\ &< M^2 + \frac{2-M^2}{2M+1}(2M+1) = 2\end{aligned}$$

AND THIS IS OUR CONTRADICTION ($M+\frac{1}{n} \in A$, BUT $M+\frac{1}{n} > M$). THUS, $M^2 \geq 2$.

TO SHOW THAT $M^2 = 2$ WE NEED ONLY SHOW THAT $M^2 > 2$ IS IMPOSSIBLE. FOR THIS WE AGAIN ARGUE BY CONTRADICTION AND SUPPOSE $M^2 > 2$.

THE IDEA NOW IS TO SHOW THAT WE CAN DECREASE M JUST A BIT AND HAVE SOMETHING WHOSE SQUARE IS STILL GREATER THAN 2, THEREBY CONTRADICTING THE FACT THAT M IS THE <u>LEAST</u> UPPER BOUND FOR A.

NOTE THAT, FOR ANY $n \in \mathbb{N}$,

$$(M-\tfrac{1}{n})^2 = M^2 - \frac{2M}{n} + \frac{1}{n^2} > M^2 - \frac{1}{n}(2M).$$

$m > M - 1$. BUT THEN $m+1 > M$ AND, SINCE $m+1 \in \mathbb{N}$, THIS CONTRADICTS THE FACT THAT M IS AN UPPER BOUND FOR $\mathbb{N}$.

$\square$

<u>COROLLARY</u>: FOR ANY REAL NUMBER $\varepsilon > 0$ THERE IS AN $n \in \mathbb{N}$ WITH $\frac{1}{n} < \varepsilon$.

<u>PROOF</u>: SUPPOSE THIS IS NOT THE CASE. THEN FOR SOME $\varepsilon > 0$, $\frac{1}{n} \geq \varepsilon$ FOR EVERY $n \in \mathbb{N}$. BUT THIS MEANS THAT $n \leq \frac{1}{\varepsilon}$ FOR EVERY $n \in \mathbb{N}$, I.E., $\frac{1}{\varepsilon}$ IS AN UPPER BOUND FOR $\mathbb{N}$ AND THIS CONTRADICTS THE PREVIOUS THEOREM. $\square$

NOW LET'S SEE WHAT COMPLETENESS CAN REALLY DO FOR US BY PROVING THAT THIS THING WE HAVE ALWAYS CALLED $\sqrt{2}$ ACTUALLY EXISTS (AS A REAL NUMBER, IF NOT AS A RATIONAL NUMBER).

WE WILL CONSIDER THE SET

$$A = \{ x \in \mathbb{R} : x^2 < 2 \}.$$

THEN $A \neq \emptyset$ (FOR EXAMPLE, $1 \in A$ SINCE $1^2 = 1 < 2$). MOREOVER, A IS BOUNDED FROM ABOVE (FOR EXAMPLE, BY 2, SINCE $x > 2 \Rightarrow x^2 > 2x > 2 \cdot 2 = 4$ SO $x \notin A$ AND THEREFORE ANY ELEMENT OF A MUST BE ≤ 2). THUS, COMPLETENESS IMPLIES THAT

$$M = \sup(A)$$

EXISTS.

WHAT WE PROPOSE TO SHOW IS THAT $M > 0$ AND $M^2 = 2$. IN OTHER WORDS, $M = \sqrt{2}$.

FIRST NOTICE THAT

$$M > 1$$

BECAUSE $1.1 \in A$ $\left((1.1)^2 = 1.21 < 2\right)$ SO $M \geq 1.1 > 1$.

NEXT WE SHOW THAT

$$M^2 \geq 2.$$

FOR THIS WE ARGUE BY CONTRADICTION AND ASSUME $M^2 < 2$.

NOTE THAT $M > 1 \Rightarrow M^2 > M$.

THE IDEA NOW IS TO SHOW THAT WE CAN INCREASE M JUST A BIT AND HAVE SOMETHING WHOSE SQUARE IS STILL LESS THAN 2 THEREBY CONTRADICTING THE FACT THAT M IS AN UPPER BOUND FOR A.

NOTE THAT, FOR ANY $n \in \mathbb{N}$,

$$\begin{aligned} \left(M + \tfrac{1}{n}\right)^2 &= M^2 + \frac{2M}{n} + \frac{1}{n^2} \\ &= M^2 + \frac{1}{n}\left(2M + \frac{1}{n}\right) \\ &\leq M^2 + \frac{1}{n}(2M + 1). \end{aligned}$$

SINCE $M > 1$ AND WE ARE ASSUMING $M^2 < 2$, $\dfrac{2-M^2}{2M+1}$ IS A POSITIVE REAL NUMBER. THUS, OUR COROLLARY ABOVE IMPLIES THAT WE CAN CHOOSE AN $n \in \mathbb{N}$ SUCH THAT

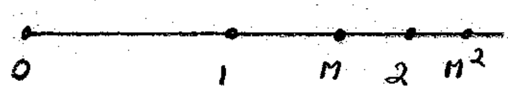
$$\frac{1}{n} < \frac{2-M^2}{2M+1}.$$

FOR SUCH AN n,

$$\begin{aligned}(M+\tfrac{1}{n})^2 &\le M^2 + \tfrac{1}{n}(2M+1) \\ &< M^2 + \frac{2-M^2}{2M+1}(2M+1) = 2\end{aligned}$$

AND THIS IS OUR CONTRADICTION ($M+\frac{1}{n} \in A$, BUT $M+\frac{1}{n} > M$). THUS, $M^2 \geq 2$.

TO SHOW THAT $M^2 = 2$ WE NEED ONLY SHOW THAT $M^2 > 2$ IS IMPOSSIBLE. FOR THIS WE AGAIN ARGUE BY CONTRADICTION AND SUPPOSE $M^2 > 2$.



THE IDEA NOW IS TO SHOW THAT WE CAN DECREASE M JUST A BIT AND HAVE SOMETHING WHOSE SQUARE IS STILL GREATER THAN 2, THEREBY CONTRADICTING THE FACT THAT M IS THE <u>LEAST</u> UPPER BOUND FOR A.

NOTE THAT, FOR ANY $n \in \mathbb{N}$,

$$(M-\tfrac{1}{n})^2 = M^2 - \frac{2M}{n} + \frac{1}{n^2} > M^2 - \tfrac{1}{n}(2M).$$

SINCE $\frac{M^2-2}{2M} > 0$ WE CAN CHOOSE AN $n \in \mathbb{N}$ FOR WHICH

$$\frac{1}{n} < \frac{M^2-2}{2M}$$

SO

$$-\frac{1}{n} > -\frac{M^2-2}{2M}$$

AND

$$\left(M-\tfrac{1}{n}\right)^2 > M^2 - \tfrac{1}{n}(2M)$$
$$> M^2 - \frac{M^2-2}{2M}(2M) = 2$$

AND WE AGAIN HAVE OUR CONTRADICTION. THUS, $M^2 = 2$.

IN A NUTSHELL,

$$\sup\left\{ x \in \mathbb{R} : x^2 < 2 \right\} = \sqrt{2} \, .$$

EXERCISES :

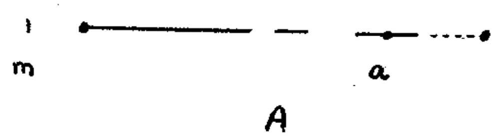
11. SHOW THAT IF A IS A NONEMPTY SUBSET OF $\mathbb{R}$ THAT IS BOUNDED FROM ABOVE, THEN $\sup(A)$ IS UNIQUE.

12. SUPPOSE THAT $A \neq \emptyset$ AND B IS BOUNDED FROM ABOVE AND $A \subseteq B$. SHOW THAT $\sup(A)$ AND $\sup(B)$ EXIST AND $\sup(A) \leq \sup(B)$.

13. LET $A = \left\{ \frac{n}{2n+1} : n \in \mathbb{N} \right\}$. SHOW THAT $\sup(A)$ EXISTS, DETERMINE ITS VALUE, AND PROVE THAT YOUR ANSWER IS CORRECT.

A NONEMPTY SUBSET A OF $\mathbb{R}$ IS SAID TO BE BOUNDED FROM BELOW IF THERE IS A REAL NUMBER m SUCH THAT

$$m \leq a \quad \text{FOR EVERY } a \in A$$



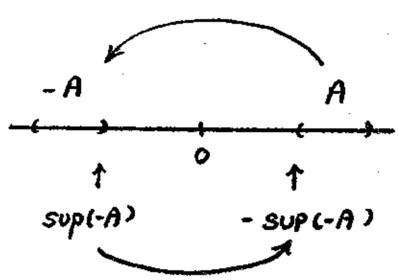
ANY SUCH NUMBER m IS CALLED A LOWER BOUND FOR A.

A LOWER BOUND m FOR A IS CALLED THE GREATEST LOWER BOUND (OR INFIMUM) OF A IF, WHENEVER m' IS A LOWER BOUND FOR A, THEN $m' \leq m$. IT IS DENOTED

$$\inf(A).$$

THEOREM: ANY NONEMPTY SUBSET A OF $\mathbb{R}$ THAT IS BOUNDED FROM BELOW HAS A GREATEST LOWER BOUND.

PROOF: THE IDEA IS TO FLIP A ACROSS THE ORIGIN TO GET A SET THAT IS BOUNDED FROM ABOVE AND SO HAS A SUPREMUM AND THEN FLIP THIS SUPREMUM BACK ACROSS THE ORIGIN.



MORE PRECISELY, WE DEFINE

$$-A = \{-a : a \in A\}.$$

NOTICE THAT

$$m \le a \ \forall a \in A \iff -a \le -m \ \forall -a \in -A$$

SO $-A$ IS BOUNDED FROM ABOVE. BY COMPLETENESS,

$$\sup(-A)$$

EXISTS.

CLAIM : $\boxed{-\sup(-A) = \inf(A)}$

TO SEE THIS FIRST NOTE THAT

$$-a \le \sup(-A) \ \forall a \in A \Rightarrow a \geq -\sup(-A) \ \forall a \in A$$

SO $-\sup(-A)$ IS A LOWER BOUND FOR A. TO SHOW THAT IT IS THE GREATEST LOWER BOUND FOR A, LET m' BE ANY OTHER LOWER BOUND FOR A. THEN

$$\begin{aligned} a \geq m' \ \forall a \in A &\Rightarrow -a \le -m' \ \forall -a \in -A \\ &\Rightarrow -m' \text{ IS AN UPPER BOUND FOR } -A \\ &\Rightarrow -m' \geq \sup(-A) \\ &\Rightarrow m' \le -\sup(-A) \\ &\Rightarrow -\sup(-A) \text{ IS THE GREATEST LOWER} \\ &\qquad \text{BOUND FOR } A \end{aligned}$$

AS REQUIRED. □

NOTE : THE EQUALITY IN THE BOX ABOVE IS WORTH REMEMBERING, AS IS THE ONE IN THE FOLLOWING EXERCISE.

EXERCISE 14 : LET A BE A NONEMPTY SUBSET OF $\mathbb{R}$ THAT IS BOUNDED FROM ABOVE. SHOW THAT INF $(-A)$ EXISTS AND

$$\sup(A) = -\inf(-A).$$

A NONEMPTY SUBSET A OF $\mathbb{R}$ IS SAID TO BE BOUNDED IF IT IS BOUNDED FROM ABOVE AND BOUNDED FROM BELOW, I.E., IF THERE EXIST CONSTANTS m AND M SUCH THAT

$$m \leq a \leq M \quad \text{FOR EVERY } a \in A.$$

IN THIS CASE, INF (A) AND SUP (A) BOTH EXIST.

EXERCISE 15 : SHOW THAT, IF A IS NONEMPTY AND BOUNDED, THEN

$$\inf(A) \leq \sup(A).$$

WHAT CAN BE SAID ABOUT A IF $\inf(A) = \sup(A)$?

BOUNDEDNESS OF A SET CAN BE REPHRASED IN TERMS OF THE ABSOLUTE VALUE FUNCTION. RECALL THAT, FOR ANY $x \in \mathbb{R}$,

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x \leq 0 \end{cases}$$

THEOREM (THE TRIANGLE INEQUALITY): $x, y \in \mathbb{R} \Rightarrow$

$$|x+y| \leq |x| + |y|.$$

<u>PROOF</u>: THE PROOF SIMPLY AMOUNTS TO CONSIDERING FOUR CASES.

(a) $x \geq 0$ AND $y \geq 0$

(b) $x \leq 0$ AND $y \leq 0$

(c) $x \leq 0$ AND $y \geq 0$

(d) $x \geq 0$ AND $y \leq 0$

HERE IS THE ARGUMENT IN CASE (c).

$$x \leq 0 \Rightarrow |x| = -x \quad \text{AND} \quad y \geq 0 \Rightarrow |y| = y$$

THUS, $x + y = -|x| + |y|$ SO

$$|x+y| = \big| -|x| + |y| \big| = \begin{cases} -|x| + |y|, & -|x|+|y| \geq 0 \\ |x| - |y|, & -|x| + |y| \leq 0 \end{cases}.$$

BUT $-|x| + |y| \leq |x| + |y|$ AND $|x| - |y| \leq |x| + |y|$ SO

$$|x+y| \leq |x| + |y|.$$

<u>EXERCISE</u> 16 : PROVE THE RESULT IN CASES (a), (b) AND (d). $\square$

WHILE ON THE SUBJECT NOTE THAT

$$|x| = \big| (x-y) + y \big| \leq |x-y| + |y|$$

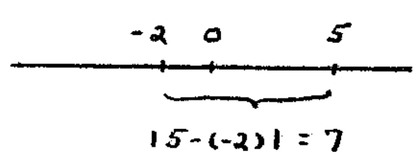
SO WE ALSO HAVE

$$\boxed{|x| - |y| \leq |x-y|}$$

WHICH WILL BE USEFUL LATER.

<u>NOTE</u> : $|x-y|$ IS CALLED THE <u>DISTANCE</u> <u>BETWEEN</u> x <u>AND</u> y.

E.G.,



EXERCISE 17 : PROVE THAT, FOR $R \geq 0$,

(a) $|x| \leq R \iff -R \leq x \leq R$

(b) A NONEMPTY SUBSET A OF $\mathbb{R}$ IS BOUNDED $\iff$ THERE IS A CONSTANT $R \geq 0$ SUCH THAT

$|a| \leq R$ FOR EVERY $a \in A$.

ESSENTIALLY EVERYTHING IN ANALYSIS IS, DIRECTLY OR INDIRECTLY, A CONSEQUENCE OF COMPLETENESS. WE WILL DERIVE ONE OF THESE CONSEQUENCES HERE AND MEET THE OTHERS AS WE PROCEED.

FIRST, A WAY OF THINKING ABOUT SUPREMA THAT IS OFTEN EASIER TO DEAL WITH. BASICALLY, IT SAYS THAT AN UPPER BOUND FOR A IS THE LEAST UPPER BOUND IF AND ONLY IF ANYTHING SMALLER FAILS TO BE AN UPPER BOUND.

THEOREM : LET A BE A NONEMPTY SUBSET OF $\mathbb{R}$. SUPPOSE M IS AN UPPER BOUND FOR A. THEN $M = \sup(A) \iff$ FOR EVERY $\varepsilon > 0$ THERE IS AN $a \in A$ WITH $a > M - \varepsilon$.

PROOF :

$\Rightarrow$ ASSUME $M = \sup(A)$. LET $\varepsilon > 0$ BE GIVEN. THEN $M - \varepsilon < M$ SO, SINCE M IS THE LEAST UPPER BOUND FOR A, $M - \varepsilon$ IS NOT AN UPPER BOUND FOR A, I.E., THERE IS AN $a \in A$ WITH $a > M - \varepsilon$.

$\Leftarrow$ NOW ASSUME THAT M IS AN UPPER BOUND FOR A AND THAT $\forall\, \varepsilon > 0$ THERE IS AN $a \in A$ WITH $a > M - \varepsilon$. TO SEE THAT M IS THE LEAST UPPER BOUND FOR A, SUPPOSE NOT. THEN THERE IS AN UPPER BOUND M' FOR A WITH $M' < M$. LET $\varepsilon = M - M'$. BY ASSUMPTION THERE IS AN $a \in A$ WITH

$$a > M - \varepsilon = M - (M - M') = M'$$

AND THIS CONTRADICTS THE FACT THAT M' IS AN UPPER BOUND FOR A. $\square$

NOW FOR THE APPLICATION.

A SUBSET D OF $\mathbb{R}$ IS SAID TO BE DENSE IN $\mathbb{R}$ IF EVERY OPEN INTERVAL (a, b), $a < b$, IN $\mathbb{R}$ CONTAINS AN ELEMENT OF D (BASICALLY, D IS EVERYWHERE YOU LOOK IN $\mathbb{R}$).

THEOREM: THE SET $\mathbb{Q}$ OF RATIONAL NUMBERS IS DENSE IN $\mathbb{R}$.

PROOF: WE BEGIN WITH A

LEMMA: FOR ANY $x \in \mathbb{R}$ THERE IS AN $n \in \mathbb{Z}$ WITH $n \le x < n+1$ ($n = [x]$ = GREATEST INTEGER $\le x$).

PROOF OF LEMMA: FIRST SUPPOSE $x \geq 0$. BY THE ARCHIMEDIAN PRINCIPLE, $\{ k \in \mathbb{N} : k > x \}$ IS NONEMPTY SO, BY THE WELL-ORDERING PRINCIPLE, THIS SET HAS A LEAST ELEMENT N. THUS, $0 \le x < N$. IF $N = 1$, THEN $0 \le x < 1$ AND WE TAKE $n = 0$. IF

$N > 1$, THEN $N-1 \in \mathbb{N}$ SO, BY THE MINIMALITY OF N, $x \geq N-1$, I.E., $N-1 \leq x < N$ SO WE TAKE $n = N-1$.

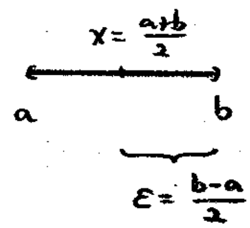
NOW SUPPOSE $x < 0$. THEN $-x > 0$ SO, BY WHAT WE HAVE JUST SHOWN, THERE IS AN $m \in \mathbb{Z}$ WITH $m \leq -x < m+1$, I.E.,

$$-m-1 < x \leq -m.$$

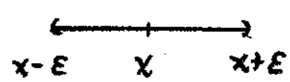
IF $x = -m$, THEN $-m \leq x < -m+1$ SO WE TAKE $n = -m$. IF $x < -m$, THEN $-m-1 < x < -m$ SO $-m-1 \leq x < -m$ AND WE TAKE $n = -m-1$. $\square$

NOW FOR THE PROOF OF THE THEOREM. LET (a,b), $a < b$, BE AN ARBITRARY OPEN INTERVAL IN $\mathbb{R}$. WE MUST SHOW THAT IT CONTAINS SOME RATIONAL NUMBER. TO SIMPLIFY THE NOTATION LET

$$\varepsilon = \frac{b-a}{2} \quad \text{AND} \quad x = \frac{a+b}{2}$$



SO THAT OUR INTERVAL IS $(x-\varepsilon, x+\varepsilon)$.



CHOOSE $n \in \mathbb{N}$ WITH $\frac{1}{n} < \varepsilon$ AND CONSIDER THE REAL NUMBER nx. BY OUR LEMMA, THERE IS AN $m \in \mathbb{Z}$ WITH

$$m \leq nx < m+1.$$

THUS,

$$\frac{m}{n} \leq x < \frac{m}{n} + \frac{1}{n} < \frac{m}{n} + \varepsilon \leq x + \varepsilon$$

SO

$$\frac{m}{n} < x + \varepsilon .$$

MOREOVER, $x < \frac{m}{n} + \frac{1}{n}$ GIVES

$$x - \frac{1}{n} < \frac{m}{n}$$

SO

$$x - \varepsilon < x - \frac{1}{n} < \frac{m}{n}$$

AND WE HAVE

$$x - \varepsilon < \frac{m}{n} < x + \varepsilon$$

AS REQUIRED. $\square$

COROLLARY : THE SET $\mathbb{R} - \mathbb{Q}$ OF IRRATIONAL NUMBERS IS DENSE IN $\mathbb{R}$.

PROOF : LET (a, b), $a < b$, BE AN OPEN INTERVAL IN $\mathbb{R}$. BY THE THEOREM WE CAN CHOOSE A RATIONAL NUMBER r_1 IN (a, b) AND THEN ANOTHER RATIONAL NUMBER r_2 IN (r_1, b). THUS, $a < r_1 < r_2 < b$.

EXERCISE 18 : SHOW THAT $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$ IS IRRATIONAL.

SINCE $r_2 - r_1 > 0$, $t > r_1$. MOREOVER,

$$t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1) < r_1 + (r_2 - r_1) = r_2$$

SO $t \in (r_1, r_2) \subseteq (a, b)$ AS REQUIRED. $\square$

ADDITIONAL PROBLEMS :

19. PROVE THAT $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ FOR ANY REAL NUMBERS $a_1, \ldots, a_n$.

20. PROVE THAT, FOR ANY REAL NUMBER $x > -1$ AND ANY $n \in \mathbb{N}$,

$$(1+x)^n \geq 1 + nx.$$

21. FIND ALL $n \in \mathbb{N}$ FOR WHICH $2n+1 < 2^n$ AND PROVE THAT YOUR ANSWER IS CORRECT.

22. FOR INTEGERS $0 \leq k \leq n$ DEFINE THE BINOMIAL COEFFICIENT $\binom{n}{k}$ BY

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

(RECALL THAT $0! = 1$).

(a) PROVE THAT $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ FOR $1 \leq k \leq n$.

(b) PROVE BY INDUCTION THAT $\binom{n}{k}$ IS ALWAYS A NATURAL NUMBER.

(c) PROVE THE BINOMIAL THEOREM: FOR $x, y \in \mathbb{R}$ AND $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

23. (a) PROVE BY INDUCTION THAT, FOR $r \neq 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

(b) PROVE THE SAME FORMULA WITHOUT INDUCTION BY WRITING $S = 1 + r + r^2 + \cdots + r^n$, MULTIPLYING THIS EQUATION BY r AND THEN SOLVING THE TWO EQUATIONS FOR S.

24. LET A BE A NONEMPTY SUBSET OF $\mathbb{R}$ THAT IS BOUNDED FROM BELOW AND LET B BE THE SET OF ALL LOWER BOUNDS FOR A. SHOW THAT B IS BOUNDED FROM ABOVE AND

$$\sup(B) = \inf(A).$$

25. FORMULATE AND PROVE AN ANALOGUE OF THE THEOREM ON PAGE 27 FOR INFIMA.

26. LET A AND B BE NONEMPTY SUBSETS OF $\mathbb{R}$ SUCH THAT $a \le b$ FOR ALL $a \in A$ AND $b \in B$.

(a) PROVE THAT $\sup(A)$ EXISTS AND $\sup(A) \le b$ FOR ALL $b \in B$.

(b) PROVE THAT $\inf(B)$ EXISTS AND $\sup(A) \le \inf(B)$.

27. LET A AND B BE NONEMPTY SUBSETS OF $\mathbb{R}$.

(a) SUPPOSE A AND B ARE BOUNDED FROM ABOVE. DEFINE

$$A+B = \{a+b : a \in A, b \in B\}.$$

PROVE THAT

$$\sup(A+B) = \sup(A) + \sup(B).$$

(b) SUPPOSE A AND B ARE BOUNDED FROM BELOW. PROVE THAT

$$\inf(A+B) = \inf(A) + \inf(B).$$

(c) SUPPOSE A IS BOUNDED FROM ABOVE AND LET k BE A POSITIVE CONSTANT. DEFINE

$$kA = \{ka : a \in A\}.$$

PROVE THAT

$$\sup(kA) = k\sup(A).$$

(d) FORMULATE AND PROVE RESULTS ANALOGOUS TO (c) FOR $k \le 0$ AND FOR INFIMA OF SETS BOUNDED FROM BELOW.

SOLUTIONS TO THE EXERCISES :

NOTE : SOLUTIONS WILL BE PROVIDED TO THE EXERCISES INCORPORATED INTO THE LECTURE NOTES, BUT NOT TO THE "ADDITIONAL PROBLEMS" AT THE END. ALL OF THE EXERCISES AND PROBLEMS SHOULD BE REGARDED AS AN INTEGRAL PART OF THE COURSE MATERIAL AND MUST BE WORKED CONSCIENTIOUSLY. THE RESULTS WILL BE USED FREELY THROUGHOUT THE COURSE.

1. SUPPOSE FIRST THAT n IS EVEN. THEN

$$1+2+3+\cdots+(n-2)+(n-1)+n =$$

$$(1+n)+(2+(n-1))+(3+(n-2))+\cdots+\left(\tfrac{n}{2}+\left(\tfrac{n}{2}+1\right)\right) =$$

$$(n+1)+(n+1)+(n+1)+\cdots+(n+1) = \frac{n}{2}(n+1)$$

$$= \frac{n(n+1)}{2}$$

IF n IS ODD, THEN $n-1$ IS EVEN SO

$$1+2+\cdots+(n-1)+n = \frac{(n-1)((n-1)+1)}{2}+n$$

$$= \frac{(n-1)n}{2}+\frac{2n}{2} = \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

AS REQUIRED.

2. $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ IS TRUE FOR $n=1$ SINCE $1^2 = \frac{1}{6}(1)(2)(3)$. NOW ASSUME THAT, FOR SOME $k \geq 1$,

$$1^2+2^2+\cdots+k^2 = \frac{1}{6}k(k+1)(2k+1).$$

THEN

$$\begin{aligned}
1^2+2^2+\cdots+k^2+(k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2 \\
&= (k+1)\left(\tfrac{1}{6}k(2k+1)+(k+1)\right) \\
&= (k+1)\tfrac{1}{6}\left(k(2k+1)+6(k+1)\right) \\
&= \tfrac{1}{6}(k+1)(2k^2+7k+6) \\
&= \tfrac{1}{6}(k+1)(k+2)(2k+3) \\
&= \tfrac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

AS REQUIRED.

3. $1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ IS TRUE FOR $n=1$ BECAUSE $1^3=\left(\frac{(1)(1+1)}{2}\right)^2$. NOW ASSUME THAT, FOR SOME $k \geq 1$,

$$1^3+2^3+\cdots+k^3=\left(\frac{k(k+1)}{2}\right)^2.$$

THEN

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \\
&= (k+1)^2\left(\frac{k^2}{4}+(k+1)\right) \\
&= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2
\end{aligned}$$

AS REQUIRED.

4. $\frac{1}{n!} > \frac{8^n}{(2n)!}$ IS EQUIVALENT TO $n! < \frac{(2n)!}{8^n}$. DOING THE ARITHMETIC SHOWS THAT THE FIRST VALUE OF n FOR WHICH THIS IS TRUE IS

$$n = 6 .$$

NOW ASSUME THAT $k! < \frac{(2k)!}{8^k}$. THEN

$$(k+1)! = k!(k+1) < \frac{(2k)!}{8^k}(k+1) = \frac{(2k)!\,4(2k+2)}{8^{k+1}}$$

$$< \frac{(2k)!\,(2k+1)(2k+2)}{8^{k+1}} = \frac{(2(k+1))!}{8^{k+1}}$$

PROVIDED $2k+1 > 4$, I.E., $k \geq 2$.

THUS, THE INEQUALITY IS SATISFIED FOR ALL

$$n \geq 6 .$$

5. ASSUME $P(n)$ IS TRUE FOR n_0 AND THAT $P(k+1)$ IS TRUE WHENEVER $P(n_0), \ldots, P(k)$ ARE ALL TRUE. LET

$$N = \{ n \in \mathbb{N} : n \geq n_0 \text{ AND } P(n) \text{ IS TRUE} \}.$$

IT IS ENOUGH TO SHOW THAT $\{n_0, n_0+1, \ldots\} \subseteq N$. SUPPOSE NOT. THEN

$$S = \{ n \in \mathbb{N} : n \geq n_0 \text{ AND } n \notin N \}$$

IS NONEMPTY. THE WELL-ORDERING PRINCIPLE IMPLIES THAT S HAS A LEAST ELEMENT s_0. $s_0 \neq n_0$ BECAUSE $P(n_0)$ IS TRUE. THUS, $s_0 > n_0$ AND SO

$s_0 - 1 \geq n_0$. SINCE s_0 IS THE LEAST ELEMENT OF S AND $n_0, \ldots, s_0 - 1$ ARE ALL $\geq n_0$, ALL OF $n_0, \ldots, s_0 - 1$ ARE IN N, I.E., $P(n_0), \ldots, P(s_0 - 1)$ ARE ALL TRUE. BUT

$$P(n_0), \ldots, P(s_0 - 1) \text{ TRUE} \Rightarrow P((s_0 - 1) + 1) \text{ IS TRUE},$$
$$\text{I.E., } P(s_0) \text{ IS TRUE}$$

AND THIS CONTRADICTS THE FACT THAT $s_0 \in S$.

6. ASSUME THAT $p_1, \ldots, p_k$ IS A LIST OF ALL OF THE PRIMES. CONSIDER THE NATURAL NUMBER $p_1 \cdots p_k + 1$. IT CANNOT BE A PRIME SINCE IT IS GREATER THAN ANY OF $p_1, \ldots, p_k$. THUS, IT MUST BE A PRODUCT OF PRIMES. BUT ANY PRODUCT OF THE $p_1, \ldots, p_k$ IS LESS THAN $p_1 \cdots p_k + 1$ SO THIS IS IMPOSSIBLE AND WE HAVE A CONTRADICTION. CONSEQUENTLY, NO FINITE LIST OF PRIMES CAN EXHAUST ALL OF THE PRIMES.

7. SUPPOSE THAT 1 AND $1'$ BOTH HAVE THE PROPERTY THAT $a \cdot 1 = 1 \cdot a = a$ AND $a \cdot 1' = 1' \cdot a = a$ FOR EVERY a. THEN, IN PARTICULAR, $1' \cdot 1 = 1'$ AND $1' \cdot 1 = 1$ SO $1' = 1$.

8. IT HAS BEEN SHOWN THAT $(-a)b = -(ab)$ FOR ALL a AND b. AND $a \cdot 0 = 0$ FOR ALL a. THUS,

$$\begin{aligned}(-a)(-b) + (-(ab)) &= (-a)(-b) + (-a)b \\ &= (-a)((-b) + b) \\ &= (-a) \cdot 0 \\ &= 0\end{aligned}$$

AND SO

$$(-a)(-b) + (-(ab)) = 0$$

$$[(-a)(-b) + (-(ab))] + ab = 0 + ab$$

$$(-a)(-b) + [-(ab) + ab] = ab$$

$$(-a)(-b) + 0 = ab$$

$$(-a)(-b) = ab$$

9. SINCE IT IS ASSUMED THAT $1 \neq 0$, #4 ON PAGE 14 GIVES $1^2 > 0$. BUT, $1^2 = 1 \cdot 1 = 1$ SO $1 > 0$. THUS, BY (P12),

$$1 + 1 > 0 + 1 = 1 > 0$$

SO, IN PARTICULAR,

$$1 + 1 \neq 0.$$

10. FOR $\mathbb{Z}_2$ THE AXIOMS (P1) - (P9) CAN BE CHECKED BY SIMPLY WRITING OUT ALL OF THE SUMS AND PRODUCTS EXPLICITLY. FOR EXAMPLE, THE TABLE FOR + GIVES

$$[0] + [0] = [0]$$

$$[0] + [1] = [1] + [0] = [1]$$

SO $[0]$ IS THE ADDITIVE IDENTITY AND THE TABLE FOR $\cdot$ GIVES

$$[1] \cdot [0] = [0] \cdot [1] = [0]$$

$$[1] \cdot [1] = [1]$$

SO $[1]$ IS THE MULTIPLICATIVE IDENTITY. NEVERTHELESS, THE TABLE FOR + GIVES

$$[1] + [1] = [0].$$

THE EXISTENCE OF ADDITIVE AND MULTIPLICATIVE INVERSES CAN

BE READ OFF DIRECTLY FROM THE TABLE, E.G.,

$$[0]+[0]=[0] \Rightarrow -[0]=[0]$$

$$[1]+[1]=[0] \Rightarrow -[1]=[1]$$

$$[1]\cdot[1]=[1] \Rightarrow [1]^{-1}=[1].$$

FOR ALL OF THE OTHER PROPERTIES ONE SIMPLY WRITES OUT ALL THE SUMS, E.G. FOR COMMUTATIVITY OF ADDITION,

$$[0]+[0]=[0]$$

$$[0]+[1]=[1]$$

$$[1]+[0]=[1]=[0]+[1]$$

$$[1]+[1]=[0].$$

11. SUPPOSE M AND M' ARE BOTH LEAST UPPER BOUNDS FOR A. THEN

M AN UPPER BOUND AND M' THE LEAST UPPER BOUND $\Rightarrow$

$$M' \leq M$$

M' AN UPPER BOUND AND M THE LEAST UPPER BOUND $\Rightarrow$

$$M \leq M'.$$

CONSEQUENTLY,

$$M' = M.$$

12. $A \neq \emptyset$ AND $A \subseteq B \Rightarrow B \neq \emptyset$. THUS, $\sup(B)$ EXISTS.

$b \leq \sup(B)\ \forall\, b \in B$ AND $A \subseteq B \Rightarrow a \leq \sup(B)\ \forall\, a \in A$.

THUS, A IS BOUNDED ABOVE SO SUP(A) EXISTS.
SINCE SUP(B) IS AN UPPER BOUND FOR A AND SUP(A) IS THE LEAST UPPER BOUND FOR A,

$$\sup(A) \leq \sup(B).$$

13. $A = \left\{ \frac{n}{2n+1} : n \in \mathbb{N} \right\}$

$A \neq \emptyset$ SINCE $\frac{1}{2(1)+1} = \frac{1}{3} \in A$.

FOR ANY $n \in \mathbb{N}$,

$$\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} < \frac{1}{2}$$

SO A IS BOUNDED FROM ABOVE BY $\frac{1}{2}$. THUS, SUP(A) EXISTS AND

$$\sup(A) \leq \frac{1}{2}.$$

WE CLAIM THAT SUP$(A) = \frac{1}{2}$. SUPPOSE NOT. THEN THERE IS AN UPPER BOUND M FOR A WITH $M < \frac{1}{2}$. BUT, FOR ANY $n \in \mathbb{N}$,

$$\frac{1}{2} - \frac{n}{2n+1} = \frac{(2n+1)-2n}{2(2n+1)}$$

$$= \frac{1}{2(2n+1)} < \frac{1}{2n+1} < \frac{1}{2n} < \frac{1}{n}$$

CHOOSE n SO THAT $\frac{1}{n} < \frac{1}{2} - M$. THEN

$$\frac{1}{2} - \frac{n}{2n+1} < \frac{1}{2} - M$$

SO

$$\frac{n}{2n+1} > M$$

AND THIS CONTRADICTS THE FACT THAT M IS AN UPPER BOUND FOR A.

14. $A \neq \emptyset$, BOUNDED FROM ABOVE.

$-A = \{-a : a \in A\}$

$a \leq \sup(A) \ \forall a \in A \Rightarrow -\sup(A) \leq -a \ \forall -a \in -A$ SO

$-A$ IS NONEMPTY AND BOUNDED FROM BELOW BY $-\sup(A)$. THUS, $\inf(-A)$ EXISTS AND

$$-\sup(A) \leq \inf(-A).$$

WE CLAIM THAT $-\sup(A) = \inf(-A)$. SUPPOSE NOT. THEN $-\sup(A) < \inf(-A)$ SO

$$-\inf(-A) < \sup(A).$$

THIS MEANS THAT $-\inf(-A)$ IS NOT AN UPPER BOUND FOR A SO THERE IS AN $a \in A$ WITH

$$a > -\inf(-A)$$

BUT THEN

$$-a < \inf(-A)$$

AND THIS CONTRADICTS THE FACT THAT $\inf(-A)$ IS A LOWER BOUND FOR $-A$. THUS, $-\sup(A) = \inf(-A)$ SO

$$\sup(A) = -\inf(-A).$$

15. $A \neq \emptyset$, BOUNDED. THEN $\inf(A)$ AND $\sup(A)$ EXIST. SUPPOSE $\inf(A) > \sup(A)$. THEN $\forall a \in A$

$$a \geq \inf(A) > \sup(A)$$

AND THIS CONTRADICTS THE FACT THAT $\sup(A)$ IS AN UPPER BOUND FOR A. THUS, $\inf(A) \leq \sup(A)$. IF $\inf(A) = \sup(A) = b$, THEN $\forall a \in A$, $a \geq b$ AND $a \leq b$ SO $a = b$, I.E., $A = \{b\}$.

16. (a) $x \geq 0$ AND $y \geq 0 \Rightarrow x+y \geq 0$ SO

$|x| = x$, $|y| = y$ AND $|x+y| = x+y$. THUS,

$$|x+y| = x+y = |x| + |y|.$$

(b) $x \leq 0$ AND $y \leq 0 \Rightarrow x+y \leq 0$ SO

$|x| = -x$, $|y| = -y$ AND $|x+y| = -(x+y)$. THUS,

$$|x+y| = -(x+y) = (-x) + (-y)$$

$$= |x| + |y|.$$

(d) $x \geq 0 \Rightarrow |x| = x$ AND $y \leq 0 \Rightarrow |y| = -y$.

THUS,

$$|x+y| = \big||x| - |y|\big|$$

$$= \begin{cases} |x| - |y| , & |x| - |y| \geq 0 \\ |y| - |x| , & |x| - |y| \leq 0 \end{cases}$$

BUT $|x| - |y| \leq |x| + |y|$ AND $|y| - |x| \leq |x| + |y|$ SO

$$|x+y| \leq |x| + |y|.$$

17. LET R BE A NONNEGATIVE CONSTANT.

(a) ASSUME $|x| \leq R$. IF $x \geq 0$, $|x| = x$ SO $0 \leq x \leq R$. IF $x \leq 0$, $|x| = -x$ SO $-x \leq R$ AND THEREFORE $-R \leq x \leq 0$. IN EITHER CASE, $-R \leq x \leq R$.

CONVERSELY, SUPPOSE $-R \leq x \leq R$ AND CONSIDER $|x|$. IF $x \geq 0$, THEN $|x| = x$ SO $-R \leq |x| \leq R$. BUT $|x| \geq 0$ SO $0 \leq |x| \leq R$. IF $x \leq 0$, THEN $|x| = -x$ SO $-R \leq x \leq R \Rightarrow R \geq -x \geq -R$, I.E. $R \geq |x| \geq -R$, OR, $-R \leq |x| \leq R$. AGAIN, $|x| \geq 0$ $\Rightarrow 0 \leq |x| \leq R$. IN ANY CASE, $|x| \leq R$.

17.(b) $A \neq \emptyset$ AND BOUNDED. THEN THERE EXIST CONSTANTS m AND M SUCH THAT

$$m \leq a \leq M$$

FOR EVERY $a \in A$. LET $R = \max\{|m|, |M|\}$. THEN $R \geq 0$ AND, FOR ANY $a \in A$,

$$a \leq M \leq |M| \leq R$$

AND

$$a \geq m \geq -|m| \geq -R$$

SO $-R \leq a \leq R$, I.E., $|a| \leq R$.

CONVERSELY, SUPPOSE THERE IS A CONSTANT $R \geq 0$ SUCH THAT $|a| \leq R$ FOR EVERY $a \in A$. THEN

$$-R \leq a \leq R$$

FOR EVERY $a \in A$ SO A IS BOUNDED ($m = -R$, $M = R$).

18. $r_1 \neq r_2$ ARE RATIONAL. WE CLAIM THAT $t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$ IS IRRATIONAL. SUPPOSE NOT. THEN

$$r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1) = r$$

FOR SOME $r \in \mathbb{Q}$. THUS,

$$\frac{1}{\sqrt{2}} = \frac{r - r_1}{r_2 - r_1}$$

AND, SINCE $r \neq r_1$,

$$\sqrt{2} = \frac{r_2 - r_1}{r - r_1}.$$

BUT $\frac{r_2 - r_1}{r - r_1}$ IS RATIONAL AND $\sqrt{2}$ IS NOT SO THIS IS A CONTRADICTION.