TOPOLOGICAL SPACES AND CONTINUOUS MAPS

SOME MOTIVATION FOR THE GENERAL DEFINITIONS : CONSIDER THE SET

$$\mathbb{R}^3 = \{ x = (x^1, x^2, x^3) \ : \ x^i \in \mathbb{R}, \ i = 1,2,3 \}$$

(EVERYTHING IS THE SAME FOR ANY $\mathbb{R}^n$).

MATHEMATICAL STRUCTURE :

1. LINEAR (VECTOR SPACE) :

$$\begin{aligned} x + y &= (x^1, x^2, x^3) + (y^1, y^2, y^3) \\ &= (x^1 + y^1, \ x^2 + y^2, \ x^3 + y^3) \\ a x &= a (x^1, x^2, x^3) \quad , \quad a \in \mathbb{R} \\ &= (a x^1, a x^2, a x^3) \end{aligned}$$

2. INNER PRODUCT :

$$\langle x, y \rangle = x^1 y^1 + x^2 y^2 + x^3 y^3$$

$$\| x \|^2 = \langle x, x \rangle = (x^1)^2 + (x^2)^2 + (x^3)^2$$

3. METRIC :

$$d(x,y) = \| y - x \|$$ = DISTANCE BETWEEN x AND y

4. TOPOLOGICAL :

$$U_\varepsilon(p) = \{ x \in \mathbb{R}^3 : d(p,x) < \varepsilon \}$$

= OPEN BALL OF RADIUS $\varepsilon > 0$ ABOUT $p \in \mathbb{R}^3$

$U \subseteq \mathbb{R}^3$ IS OPEN IN $\mathbb{R}^3$ IF $\forall \, p \in U$ $\exists \, \varepsilon > 0$ S.T. $U_\varepsilon(p) \subseteq U$

EXERCISE 1 : SHOW THAT THE COLLECTION OF OPEN SETS IN $\mathbb{R}^3$ HAS THE FOLLOWING PROPERTIES :

(a) $\emptyset$ AND $\mathbb{R}^3$ ARE OPEN IN $\mathbb{R}^3$

(b) U_α OPEN IN $\mathbb{R}^3$ $\forall \alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha$ OPEN IN $\mathbb{R}^3$

(c) $U_1, \dots, U_k$ OPEN IN $\mathbb{R}^3 \Rightarrow U_1 \cap \dots \cap U_k$ OPEN IN $\mathbb{R}^3$

RECALL THE FOLLOWING DEFINITION FROM ANALYSIS : A MAP $f : \mathbb{R}^3 \to \mathbb{R}^3$ IS CONTINUOUS IF FOR EACH $p \in \mathbb{R}^3$ AND EACH $\varepsilon > 0$ $\exists \delta > 0$ S.T.

$$d(p,x) < \delta \quad \Longrightarrow \quad d(f(p), f(x)) < \varepsilon$$

THEOREM : $f : \mathbb{R}^3 \to \mathbb{R}^3$ IS CONTINUOUS IFF INVERSE IMAGES OF OPEN SETS ARE OPEN, I.E., IFF $\forall$ OPEN V IN $\mathbb{R}^3$,

$$f^{-1}(V) = \{ x \in \mathbb{R}^3 : f(x) \in V \}$$

IS OPEN IN $\mathbb{R}^3$.

PROOF : $\Rightarrow$ ASSUME f IS CONTINUOUS. LET V BE OPEN IN $\mathbb{R}^3$. WE SHOW THAT $f^{-1}(V)$ IS OPEN IN $\mathbb{R}^3$.

LET $p \in f^{-1}(V)$. THEN $f(p) \in V$. SINCE V IS OPEN IN $\mathbb{R}^3$ $\exists \varepsilon > 0$ S.T. $U_\varepsilon(f(p)) \subseteq V$.

SINCE f IS CONTINUOUS $\exists\, \delta > 0$ S.T.

$$d(p, x) < \delta \implies d(f(p), f(x)) < \varepsilon$$

I.E.,

$$x \in U_\delta(p) \implies f(x) \in U_\varepsilon(f(p)) \subseteq V$$

I.E.,

$$U_\delta(p) \subseteq f^{-1}(V)$$

AS REQUIRED.

$\Leftarrow$ NOW ASSUME f SATISFIES V OPEN $\Rightarrow f^{-1}(V)$ OPEN

WE SHOW THAT f IS CONTINUOUS. LET $p \in \mathbb{R}^3$ AND SUPPOSE $\varepsilon > 0$ IS GIVEN.

$$U_\varepsilon(f(p)) \text{ OPEN} \implies f^{-1}(U_\varepsilon(f(p))) \text{ OPEN}$$

SINCE $p \in f^{-1}(U_\varepsilon(f(p)))$, $\exists\, \delta > 0$ S.T. $U_\delta(p) \subseteq f^{-1}(U_\varepsilon(f(p)))$.

THUS,

$$x \in U_\delta(p) \implies f(x) \in U_\varepsilon(f(p))$$

I.E.,

$$d(p, x) < \delta \implies d(f(p), f(x)) < \varepsilon$$

AS REQUIRED. $\square$

THE POINT HERE IS THAT "CONTINUITY" CAN BE EXPRESSED ENTIRELY IN TERMS OF "OPEN SETS".

GENERALIZATION MOTIVATED BY THIS :

A <u>TOPOLOGY</u> ON A NONEMPTY SET X IS A COLLECTION $\mathcal{T}_X$ OF SUBSETS OF X WITH THE FOLLOWING PROPERTIES :

(a) $\emptyset$ AND X ARE IN $\mathcal{T}_X$

(b) $U_\alpha \in \mathcal{T}_X \ \forall \alpha \in \mathcal{A} \Rightarrow \bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}_X$

(c) $U_1, \dots, U_k \in \mathcal{T}_X \Rightarrow U_1 \cap \cdots \cap U_k \in \mathcal{T}_X$

THE SETS IN $\mathcal{T}_X$ ARE CALLED THE <u>OPEN SETS</u> OF THE TOPOLOGY AND X TOGETHER WITH $\mathcal{T}_X$ IS CALLED A <u>TOPOLOGICAL SPACE</u>.

IF $\mathcal{T}_X$ IS A TOPOLOGY FOR X AND $\mathcal{T}_Y$ IS A TOPOLOGY FOR Y, THEN A MAPPING

$$f : X \to Y$$

IS SAID TO BE <u>CONTINUOUS</u> IF

$$V \in \mathcal{T}_Y \Rightarrow f^{-1}(V) \in \mathcal{T}_X .$$

<u>EXAMPLES</u> :

1. THE COLLECTION OF OPEN SETS IN $\mathbb{R}^3$ DESCRIBED ABOVE IS THE <u>USUAL</u> (OR <u>EUCLIDEAN</u>) <u>TOPOLOGY</u> FOR $\mathbb{R}^3$.

IN EXACTLY THE SAME WAY ONE OBTAINS THE <u>USUAL</u> (OR <u>EUCLIDEAN</u>) <u>TOPOLOGY</u> ON ANY $\mathbb{R}^n$.

THE SAME PROOF AS THAT GIVEN FOR $\mathbb{R}^3$ ABOVE SHOWS THAT ANY MAP

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

IS "CONTINUOUS WITH RESPECT TO THE USUAL TOPOLOGIES ON $\mathbb{R}^n$ AND $\mathbb{R}^m$" (V OPEN IN $\mathbb{R}^m \Rightarrow f^{-1}(V)$ OPEN IN $\mathbb{R}^n$) IF AND ONLY IF IT IS "CONTINUOUS IN THE SENSE OF ANALYSIS" ($\forall\, p \in \mathbb{R}^n$ AND $\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ S.T. $d(p,x) < \delta \Rightarrow d(f(p), f(x)) < \varepsilon$) THUS, ANYTHING WE KNOW FROM ANALYSIS TO BE CONTINUOUS, E.G.,

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$f(x^1, x^2, x^3) = \left(\frac{x^1}{1+(x^3)^2}, \frac{x^2}{1+(x^3)^2} \right)$$

IS CONTINUOUS IN OUR NEW SENSE AS WELL.

<u>EXERCISE</u> 2 : ANY X CAN BE GIVEN THE <u>DISCRETE</u> <u>TOPOLOGY</u> IN WHICH EVERY SUBSET OF X IS OPEN (I.E., $\mathcal{T}_X$ IS THE COLLECTION OF ALL SUBSETS OF X). VERIFY THAT THIS REALLY IS A TOPOLOGY. NOW, GIVE $\mathbb{R}^3$ THE DISCRETE TOPOLOGY AND GIVE $\mathbb{R}^2$ ITS USUAL TOPOLOGY AND FIND <u>ALL</u> OF THE CONTINUOUS MAPS $f : \mathbb{R}^3 \to \mathbb{R}^2$.

2. IF X' IS A TOPOLOGICAL SPACE WITH TOPOLOGY $\mathcal{T}_{X'}$, AND X IS ANY NONEMPTY SUBSET OF X', THEN X INHERITS A <u>RELATIVE</u> (OR <u>SUBSPACE</u>) <u>TOPOLOGY</u> $\mathcal{T}_X$ FROM X' DEFINED BY

$$\mathcal{T}_X = \{ U = X \cap U' \ : \ U' \in \mathcal{T}_{X'} \}$$

<u>NOTE</u> : THIS REALLY IS A TOPOLOGY FOR X SINCE

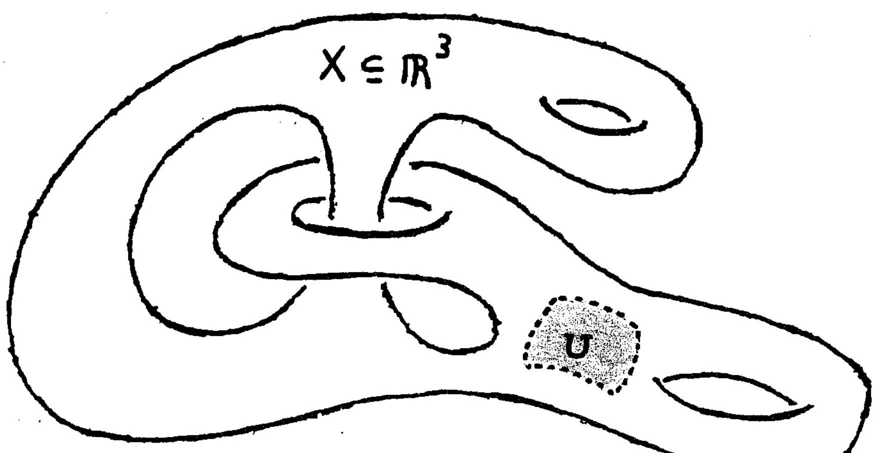
$$X \cap \emptyset = \emptyset$$

$$X \cap X' = X$$

$$\bigcup_{\alpha \in A} (X \cap U'_\alpha) = X \cap (\bigcup_{\alpha \in A} U'_\alpha)$$

$$(X \cap U'_1) \cap \cdots \cap (X \cap U'_k) = X \cap (U'_1 \cap \cdots \cap U'_k)$$

E.G., IF $X \subseteq \mathbb{R}^3$ AND $\mathbb{R}^3$ HAS ITS USUAL TOPOLOGY, RELATIVE OPEN SETS U IN X LOOK LIKE



A SUBSET X OF X' WITH THE RELATIVE TOPOLOGY IS CALLED A (<u>TOPOLOGICAL</u>) <u>SUBSPACE</u> OF X'

ANY SUBSET OF ANY EUCLIDEAN SPACE $\mathbb{R}^n$ CAN THEREFORE BE REGARDED AS A TOPOLOGICAL SPACE, E.G., THE <u>n-SPHERE</u>

$$S^n = \{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots + (x^{n+1})^2 = 1 \}$$

OTHER IMPORTANT EXAMPLES OF THIS TYPE ARE VARIOUS SETS OF MATRICES.

<u>NOTE</u>: ANY $m \times n$ REAL MATRIX CAN BE VIEWED AS AN mn-TUPLE OF REAL NUMBERS:

$$\begin{pmatrix} a^{11} & \cdots & a^{1n} \\ \vdots & & \vdots \\ a^{m1} & \cdots & a^{mn} \end{pmatrix} \longrightarrow (a^{11}, \ldots, a^{1n}, \ldots, a^{m1}, \ldots, a^{mn})$$

SO ANY SET OF REAL MATRICES HAS A TOPOLOGY AS A SUBSPACE OF SOME EUCLIDEAN SPACE.

E.G.,

<u>GENERAL LINEAR GROUP</u>: $GL(n, \mathbb{R})$ = ALL $n \times n$ MATRICES A THAT ARE NONSINGULAR (INVERTIBLE)

$$= \{ A = (a^{ij})_{i,j=1,\ldots,n} : a^{ij} \in \mathbb{R}, \ \det A \neq 0 \}$$

IS A TOPOLOGICAL SUBSPACE OF $\mathbb{R}^{n^2}$.

NOTE THAT det , THOUGHT OF AS A REAL-VALUED FUNCTION ON $\mathbb{R}^{n^2}$ (I.E., ON $n \times n$ REAL MATRICES) IS JUST A POLYNOMIAL IN THE ENTRIES (I.E., COORDINATES) SO IT IS CONTINUOUS. THUS,

$$GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$$

IS ACTUALLY AN <u>OPEN</u> SUBSPACE OF $\mathbb{R}^{n^2}$.

<u>ORTHOGONAL</u> <u>GROUP</u> : $O(n)$ = ALL $n \times n$ ORTHOGONAL MATRICES, I.E., THOSE A SATISFYING

$$A^T A = A A^T = id_{n \times n}$$

<u>NOTE</u> : THOUGHT OF AS LINEAR TRANSFORMATIONS $A : \mathbb{R}^n \to \mathbb{R}^n$ THESE ARE PRECISELY THOSE WHICH PRESERVE THE USUAL INNER PRODUCT $\langle \, , \, \rangle$:

$$\langle Ax, Ay \rangle = \langle x, y \rangle$$

ALSO NOTE THAT

$$\det(AA^T) = \det(id_{n \times n})$$

$$\det(A)\det(A^T) = 1$$

$$(\det A)^2 = 1$$

$$\det A = \pm 1$$

<u>SPECIAL ORTHOGONAL GROUP</u> : $SO(n) = \{A \in O(n) : \det A = 1\}$

$O(n)$ AND $SO(n)$ ARE BOTH TOPOLOGICAL SUBSPACES OF $\mathbb{R}^{n^2}$ AND WE CLAIM THAT $SO(n)$ IS ACTUALLY AN <u>OPEN</u> SET IN $O(n)$.

<u>LEMMA</u> : LET X BE A SUBSPACE OF X' AND LET $f : X' \to Y$ BE A CONTINUOUS MAP. THEN THE RESTRICTION

$$f|X : X \to Y$$

OF f TO X IS CONTINUOUS.

<u>PROOF</u> : LET V BE AN OPEN SET IN Y. SINCE f IS CONTINUOUS, $f^{-1}(V)$ IS OPEN IN X'. THUS, $X \cap f^{-1}(V)$ IS OPEN IN X. BUT $X \cap f^{-1}(V) = (f|X)^{-1}(V)$ SO $f|X$ IS CONTINUOUS. $\square$

THUS, THE RESTRICTION OF $\det$ TO $O(n)$ IS CONTINUOUS. SINCE $SO(n)$ IS THE INVERSE IMAGE UNDER THIS CONTINUOUS MAP OF $(0,\infty) \subseteq \mathbb{R}$, IT IS OPEN IN $O(n)$.

<u>EXERCISE</u> 3 : LET Y BE A SUBSPACE OF Y' AND $f : X \to Y'$ A CONTINUOUS MAP WITH $f(X) \subseteq Y$. SHOW THAT, THOUGHT OF AS A MAP $f : X \to Y$ FROM X TO Y, f IS CONTINUOUS.

EXERCISE 4 : LET Y BE A SUBSPACE OF Y' AND $f : X \to Y$ A CONTINUOUS MAP. SHOW THAT, THOUGHT OF AS A MAP $f : X \to Y'$, f IS CONTINUOUS.

EXERCISE 5 : LET $f : X \to Y$ AND $g : Y \to Z$ BE CONTINUOUS MAPS. SHOW THAT THE COMPOSITION $g \circ f : X \to Z$ IS CONTINUOUS.

3. THE STANDARD TOPOLOGY ON $\mathbb{R}^n$ IS DEFINED IN TERMS OF THE "DISTANCE" BETWEEN POINTS IN $\mathbb{R}^n$ (DISTANCE $\rightarrow$ OPEN BALLS $\rightarrow$ OPEN SETS). THIS IS EASY TO GENERALIZE :

LET X BE A SET. A METRIC ON X IS A FUNCTION

$$d : X \times X \to \mathbb{R}$$

THAT SATISFIES

(a) $d(y,x) = d(x,y) \quad \forall\, x,y \in X$

(b) $d(x,y) \geq 0$ AND $d(x,y) = 0$ IFF $x = y$

(c) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in X$

THE PAIR (X,d) IS CALLED A METRIC SPACE. GIVEN ANY $p \in X$ AND ANY $\varepsilon > 0$ WE DEFINE

$$U_\varepsilon^d(p) \ = \ \text{OPEN } \underline{d}\text{-}\underline{\text{BALL}} \text{ OF RADIUS } \varepsilon \text{ ABOUT } p$$

$$= \{ x \in X : d(p,x) < \varepsilon \}$$

NOW DEFINE A SET $U \subseteq X$ TO BE OPEN IF $\forall p \in U \ \exists \varepsilon > 0$ S.T. $U_\varepsilon^d(p) \subseteq U$.

<u>EXERCISE</u> 6 : SHOW THAT THIS DEFINES A TOPOLOGY ON X (CALLED THE <u>METRIC</u> <u>TOPOLOGY</u>).

AS A CONCRETE EXAMPLE, CONSIDER THE SET

$$C[0,1] \ = \ \text{ALL CONTINUOUS, REAL-VALUED FUNCTIONS ON } [0,1]$$

FOR $f, g \in C[0,1]$ DEFINE

$$d(f,g) = \max \{ |f(x) - g(x)| : 0 \le x \le 1 \}$$

<u>EXERCISE</u> 7 : SHOW THAT THIS REALLY DOES DEFINE A METRIC ON $C[0,1]$.

<u>EXERCISE</u> 8 : DESCRIBE (GEOMETRICALLY) THE OPEN ε-BALL ABOUT $f \in C[0,1]$ (DRAW A PICTURE).

<u>EXERCISE</u> 9 : LET $\ell_\infty(\mathbb{R})$ DENOTE THE SET OF ALL <u>BOUNDED</u> SEQUENCES

$$x = \{x_n\}_{n=1}^{\infty} = \{x_1, x_2, x_3, \dots\}$$

OF REAL NUMBERS. DEFINE

$$d(x,y) = d(\{x_n\}, \{y_n\}) = \sup_n |y - x_n|$$

SHOW THAT THIS DEFINES A METRIC ON $\ell_\infty(\mathbb{R})$.

<u>EXERCISE</u> 10 : A SUBSET C OF A TOPOLOGICAL SPACE X IS SAID TO BE <u>CLOSED</u> IN X IF ITS COMPLEMENT $X - C$ IS OPEN IN X. SHOW THAT

(a) $\emptyset$ AND X ARE CLOSED IN X

(b) C_α CLOSED IN X $\forall \alpha \in \mathcal{A} \Rightarrow \bigcap_{\alpha \in \mathcal{A}} C_\alpha$ CLOSED IN X

(c) $C_1, \dots, C_k$ CLOSED IN $X \Rightarrow C_1 \cup \dots \cup C_k$ CLOSED IN X.

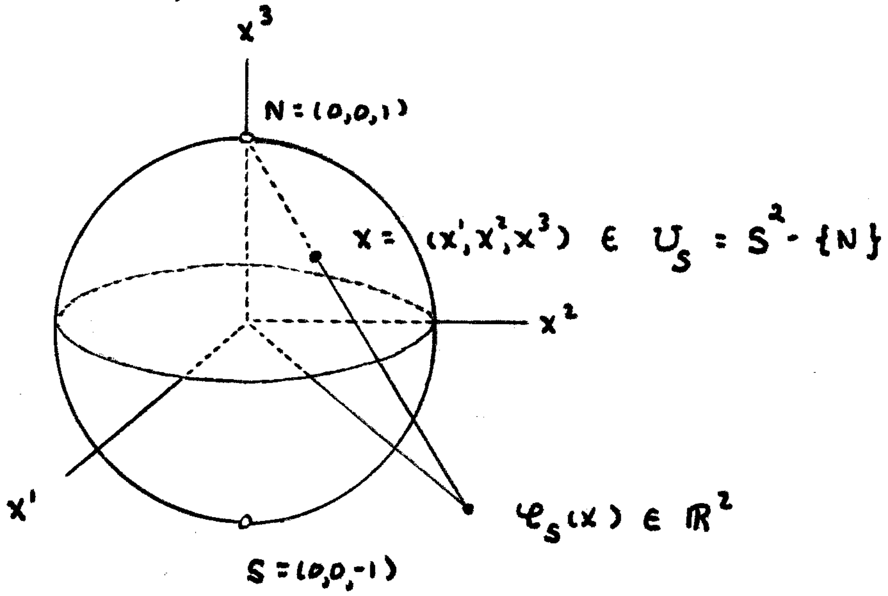
<u>EXERCISE</u> 11 : SHOW THAT $f : X \to Y$ IS CONTINUOUS IFF C CLOSED IN $Y \Rightarrow f^{-1}(C)$ CLOSED IN X.

<u>EXERCISE</u> 12 : DESCRIBE AT LEAST ONE "INTERESTING" EXAMPLE OF A CLOSED SET IN EACH OF $C[0,1]$ AND $\ell_\infty(\mathbb{R})$.

TO MOTIVATE THE NEXT IMPORTANT IDEA WE COMPUTE AN EXAMPLE.

$$S^2 = \{ x = (x^1, x^2, x^3) \in \mathbb{R}^3 : \|x\|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$$

(WITH ITS RELATIVE TOPOLOGY AS A SUBSPACE OF $\mathbb{R}^3$).



DEFINE

$$U_S = S^2 - \{N\} \qquad \text{(AN OPEN SET IN } S^2\text{)}$$

$$\varphi_S : U_S \to \mathbb{R}^2 \qquad \text{(\underline{STEREOGRAPHIC} \underline{PROJECTION} \underline{FROM} \underline{N})}$$

$\varphi_S(x^1, x^2, x^3)$ = INTERSECTION WITH $x^3 = 0$ OF THE STRAIGHT LINE THROUGH $N = (0,0,1)$ AND $x = (x^1, x^2, x^3)$

$$= \left(\frac{x^1}{1 - x^3} , \frac{x^2}{1 - x^3} \right)$$

<u>EXERCISE</u> 13 : PROVE THIS LAST FORMULA.

φ_S IS CONTINUOUS ON U_S (BECAUSE IT IS THE RESTRICTION TO U_S OF A (CALCULUS) CONTINUOUS MAP ON $\mathbb{R}^3 - \{ x^3 = 1 \}$).

φ_S IS ALSO <u>ONE-TO-ONE</u> ON U_S AND MAPS <u>ONTO</u> $\mathbb{R}^2$ BECAUSE ITS INVERSE IS EASILY FOUND (INTERSECT THE LINE THROUGH N AND $y = (y^1, y^2) \in \mathbb{R}^2$ WITH S^2):

$$\varphi_S^{-1} : \mathbb{R}^2 \to U_S$$

$$\varphi_S^{-1}(y) = \varphi_S^{-1}(y^1, y^2) = \frac{1}{1+\|y\|^2}\left(2y^1, 2y^2, \|y\|^2 - 1\right)$$

φ_S^{-1} IS ALSO CONTINUOUS.

φ_S "IDENTIFIES" U_S WITH $\mathbb{R}^2$

φ_S "COORDINATIZES" U_S, E.G., $S = (0,0,-1)$ IS ASSIGNED COORDINATES $(0,0)$ BY φ_S.

SIMILARLY, ONE CAN DEFINE <u>STEREOGRAPHIC</u> <u>PROJECTION</u> <u>FROM</u> <u>S</u>:

$$\varphi_N : U_N = S^2 - \{S\} \to \mathbb{R}^2$$

$$\varphi_N(x) = \varphi_N(x^1, x^2, x^3) = \left(\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3}\right)$$

$$\varphi_N^{-1} : \mathbb{R}^2 \to U_N$$

$$\varphi_N^{-1}(y) = \varphi_N^{-1}(y^1, y^2) = \frac{1}{1+\|y\|^2}\left(2y^1, 2y^2, 1 - \|y\|^2\right)$$

GENERAL DEFINITIONS MOTIVATED BY THIS :

IF X AND Y ARE TOPOLOGICAL SPACES AND IF THERE EXISTS A CONTINUOUS BIJECTION $h : X \to Y$ OF X ONTO Y WITH A CONTINUOUS INVERSE $h^{-1} : Y \to X$, THEN h IS CALLED A <u>HOMEOMORPHISM</u> (AS IS h^{-1}) AND X AND Y ARE SAID TO BE <u>HOMEOMORPHIC</u> (OR <u>TOPOLOGICALLY EQUIVALENT</u>) AND WE WRITE $X \cong Y$.

E.G., U_S AND U_N ARE BOTH HOMEOMORPHIC TO $\mathbb{R}^2$ (HOWEVER, WE WILL SEE LATER THAT S^2 IS <u>NOT</u> HOMEOMORPHIC TO $\mathbb{R}^2$).

<u>EXERCISE</u> 14 : SHOW THAT $\cong$ IS AN EQUIVALENCE RELATION ON THE COLLECTION OF ALL TOPOLOGICAL SPACES, I.E.,

(a) $X \cong X$

(b) $X \cong Y \Rightarrow Y \cong X$

(c) $X \cong Y$ AND $Y \cong Z \Rightarrow X \cong Z$

IF X IS A TOPOLOGICAL SPACE AND n IS A POSITIVE INTEGER, THEN AN <u>n-DIMENSIONAL</u> (<u>COORDINATE</u>) <u>CHART</u> ON X IS A PAIR (U, φ), WHERE U IS AN OPEN SET IN X AND φ IS A HOMEOMORPHISM OF U ONTO AN OPEN SET $\varphi(U)$ IN $\mathbb{R}^n$.

E.G., (U_S, φ_S) AND (U_N, φ_N) ARE
2-DIMENSIONAL CHARTS ON S^3.

IF $x \in U$, THEN (U, φ) IS CALLED A <u>CHART AT</u> x. IF THERE IS A CHART AT x FOR EVERY $x \in X$, THEN X IS SAID TO BE <u>LOCALLY EUCLIDEAN</u>.

E.G., S^2 IS LOCALLY EUCLIDEAN.

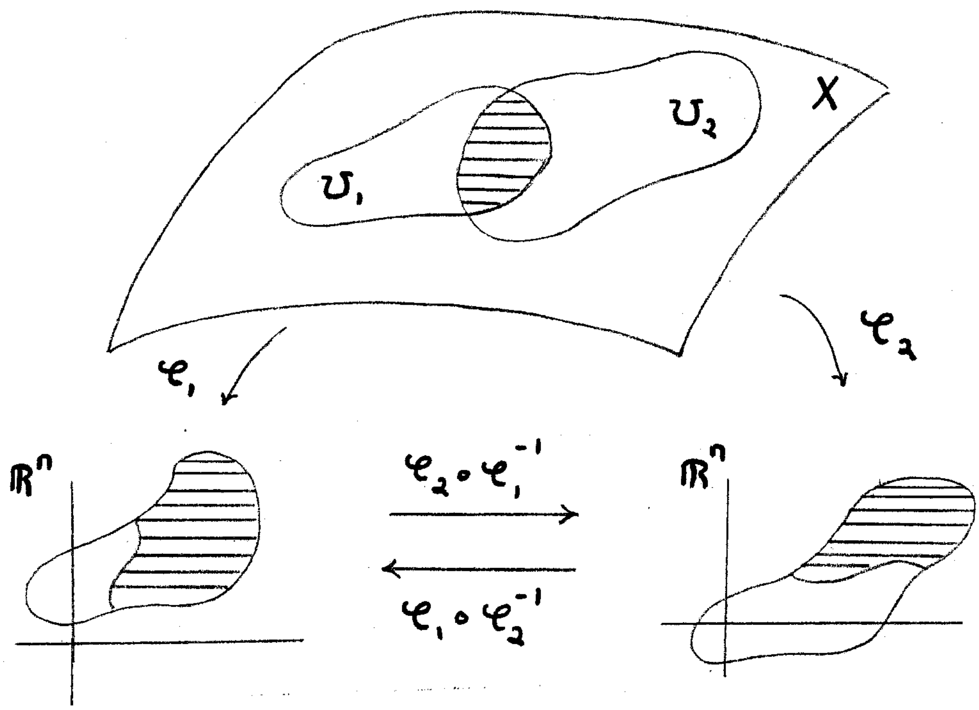
NOTE THAT, FOR POINTS X IN $U_S \cap U_N = S^2 - \{N, S\}$, TWO SETS OF COORDINATES HAVE BEEN SUPPLIED, E.G.,

$$\varphi_S\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}, \tfrac{1}{2}\right) = (\sqrt{2}, 1)$$

$$\varphi_N\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{2}, \tfrac{1}{2}\right) = \left(\tfrac{\sqrt{2}}{3}, \tfrac{1}{3}\right)$$

WE WILL NEED THE "COORDINATE TRANSFORMATIONS" THAT RELATE THESE TWO.

GENERAL DEFINITION: IF (U_1, φ_1) AND (U_2, φ_2) ARE TWO n-DIMENSIONAL CHARTS ON X AND $U_1 \cap U_2 \neq \phi$, THEN THE <u>OVERLAP MAPS</u> (OR <u>COORDINATE TRANSFORMATIONS</u>) ARE



$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$$

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \longrightarrow \varphi_1(U_1 \cap U_2)$$

THESE ARE (INVERSE) HOMEOMORPHISMS THAT RELATE THE TWO SETS OF COORDINATES PROVIDED POINTS OF $U_1 \cap U_2$.

LET'S COMPUTE THE OVERLAP MAPS FOR (U_S, φ_S) AND (U_N, φ_N) ON S^2:

$$U_S \cap U_N = S^2 - \{N, S\}$$

$$\varphi_S(U_S \cap U_N) = \varphi_N(U_S \cap U_N) = \mathbb{R}^2 - \{(0,0)\}$$

$$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 - \{(0,0)\}$$

$$\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2 - \{(0,0)\}$$

E.G., FOR $y = (y^1, y^2) \in \mathbb{R}^2 - \{(0,0)\}$

$$(\varphi_S \circ \varphi_N^{-1})(y) = \varphi_S(\varphi_N^{-1}(y^1, y^2))$$

$$= \varphi_S \left(\frac{2y^1}{1+\|y\|^2}, \frac{2y^2}{1+\|y\|^2}, \frac{1-\|y\|^2}{1+\|y\|^2} \right)$$

$$= \left(\frac{\dfrac{2y^1}{1+\|y\|^2}}{1 - \dfrac{1-\|y\|^2}{1+\|y\|^2}}, \frac{\dfrac{2y^2}{1+\|y\|^2}}{1 - \dfrac{1-\|y\|^2}{1+\|y\|^2}} \right)$$

$$= \cdots \quad \text{(SOME ALGEBRA)}$$

$$= \left(\frac{y^1}{\|y\|^2}, \frac{y^2}{\|y\|^2} \right) = \frac{1}{\|y\|^2} y$$

SIMILARLY,

$$(\varphi_N \circ \varphi_S^{-1})(y) = \frac{1}{\|y\|^2} y$$

EXERCISE 15 : GENERALIZE ALL OF THIS TO THE n-SPHERE, I.E., DEFINE STEREOGRAPHIC PROJECTION CHARTS (U_S, φ_S) AND (U_N, φ_N) FROM THE NORTH AND SOUTH POLES OF S^n AND FIND THE OVERLAP MAPS.

EXERCISE 16 : DEFINE ANOTHER CHART (U, φ) ON S^2 BY "PROJECTING TO OPEN UPPER HEMISPHERE VERTICALLY DOWN", I.E.,

$$U = \{ (x^1, x^2, x^3) \in S^2 : x^3 > 0 \}$$

$$\varphi : U \longrightarrow \mathbb{R}^2$$

$$\varphi(x^1, x^2, x^3) = (x^1, x^2)$$

SHOW THAT THIS REALLY IS A CHART ON S^2 AND THEN COMPUTE ITS OVERLAP MAPS WITH (U_S, φ_S), BEING PARTICULARLY CAREFUL ABOUT THE DOMAINS OF $\varphi_S \circ \varphi^{-1}$ AND $\varphi \circ \varphi_S^{-1}$.

EXERCISE 17 : PROVE THAT EVERY OPEN INTERVAL IN $\mathbb{R}$ IS HOMEOMORPHIC TO $\mathbb{R}$.

EXERCISE 18 : PROVE THAT EVERY d-OPEN BALL $U_\varepsilon^d(p)$ IN A METRIC SPACE (X, d) IS ACTUALLY AN OPEN SET IN THE METRIC TOPOLOGY.