TRIGONOMETRIC SUBSTITUTION

TRIGONOMETRIC SUBSTITUTION : USEFUL IDENTITIES :

$$1 - \sin^2\theta \qquad \cos^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

OR, MORE GENERALLY,

$$a^2 - (a\sin\theta)^2 = (a\cos\theta)^2$$

$$a^2 + (a\tan\theta)^2 = (a\sec\theta)^2$$

$$(a\sec\theta)^2 - a^2 = (a\tan\theta)^2$$

NOW CONSIDER AN INTEGRAL SUCH AS

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx \qquad \underline{\text{NOTE}} : -2 < x < 2$$

THE IDEA IS TO INTRODUCE A NEW VARIABLE θ THAT WILL TURN $4 - x^2$ INTO A PERFECT SQUARE, THUS GETTING RID OF THE SQUARE ROOT.

WANT :

$$4 - x^2 = (\quad)^2$$

$$2^2 - x^2 = (\quad)^2$$

LOOKS LIKE :

$$2^2 - (2\sin\theta)^2 = (2\cos\theta)^2$$

SUGGESTS CHOOSING

$$x = 2\sin\theta$$

THEN

$$\sqrt{4-x^2} = \sqrt{2^2 - x^2} = \sqrt{2^2 - (2\sin\theta)^2}$$

$$= \sqrt{(2\cos\theta)^2}$$

$$= 2\cos\theta$$

NOTE : MUST ASSUME $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$dx = 2\cos\theta \, d\theta$$

THUS,

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{(2\sin\theta)^2(2\cos\theta)}(2\cos\theta\,d\theta)$$

$$= \frac{1}{4}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$$

$$= -\frac{1}{4}\frac{\frac{1}{2}\sqrt{4-x^2}}{\frac{1}{2}x} + C = -\frac{\sqrt{4-x^2}}{4x} + C$$

MORE EXAMPLES :

1. $\int \frac{dx}{\sqrt{1+x^2}}$

NOTE : IF YOU HAPPEN TO REMEMBER THE INVERSE HYPERBOLIC FUNCTIONS, THEN YOU KNOW THAT

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C = \ln | x + \sqrt{x^2+1} | + C$$

IF YOU DON'T, THEN DO A TRIGONOMETRIC SUBSTITUTION AS FOLLOWS :

$$1 + x^2 = (\quad)^2$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$x = \tan\theta$$

$$\sqrt{1+x^2} = \sec\theta \qquad \text{NOTE} : -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

$$dx = \sec^2\theta \, d\theta$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta \, d\theta}{\sec\theta} = \int \sec\theta \, d\theta =$$

$$\ln | \sec\theta + \tan\theta | + C = \ln | \sqrt{1+x^2} + x | + C$$

2. $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$ NOTE: ASSUME $x \geq 5$

$x^2 - 5^2 = (\quad)^2$

$(5\sec\theta)^2 - 5^2 = (5\tan\theta)^2$

$x = 5\sec\theta$

$\sqrt{x^2-25} = 5\tan\theta$ NOTE: ASSUME $0 \leq \theta < \frac{\pi}{2}$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta =$$

$$5\int \tan^2\theta\,d\theta = 5\int(\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C =$$

$$\sqrt{x^2-25} - 5\arctan\left(\frac{\sqrt{x^2-25}}{5}\right) + C$$

NOTE: COULD WRITE $\theta = \operatorname{arcsec}\frac{x}{5}$
BUT IT IS GENERALLY BEST TO STICK
WITH THE INVERSES OF SIN, COS AND TAN.

3. $\displaystyle\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx$

THERE ARE TWO OPTIONS FOR DOING A DEFINITE INTEGRAL:

A. INDEFINITE INTEGRAL FIRST AND USE ORIGINAL LIMITS:

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \cdots = \sqrt{x^2-25} - 5\arctan\left(\frac{\sqrt{x^2-25}}{5}\right) + C$$

SO

$$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25}\,\Big|_5^{10} - 5\arctan\left(\frac{\sqrt{x^2-25}}{5}\right)\Big|_5^{10}$$

$$= (5\sqrt{3} - 0) - 5\left(\arctan(\sqrt{3}) - \arctan 0\right)$$

$$= 5\sqrt{3} - 5\left(\frac{\pi}{3} - 0\right)$$

$$= 5\sqrt{3} - \frac{5\pi}{3}$$

B. CHANGE THE LIMITS WHILE MAKING THE SUBSTITUTION:

$$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = 5\int_0^{\frac{\pi}{3}} \tan^2\theta\,d\theta = \cdots$$

$$= 5\tan\theta\,\Big|_0^{\frac{\pi}{3}} - 5\theta\,\Big|_0^{\frac{\pi}{3}}$$

$$= 5\sqrt{3} - \frac{5\pi}{3}$$

$x = 5\sec\theta$

$x = 5 \Rightarrow \sec\theta = 1$

$\Rightarrow \theta = 0$

$x = 10 \Rightarrow \sec\theta = 2$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

4. $\displaystyle\int \frac{x^2}{\sqrt{9+x^2}}\,dx$

$9 + x^2 = (\quad)^2$

$3^2 + x^2 = (\quad)^2$

$3^2 + (3\tan\theta)^2 = (3\sec\theta)^2$

$x = 3\tan\theta$

$\sqrt{9+x^2} = 3\sec\theta$ <u>NOTE</u>: ASSUME $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 3\sec^2\theta\,d\theta$

$$\int \frac{x^2}{\sqrt{9+x^2}}\,dx = \int \frac{9\tan^2\theta}{3\sec\theta}(3\sec^2\theta\,d\theta) = 9\int \tan^2\theta\sec\theta\,d\theta =$$

$$9\int(\sec^2\theta - 1)\sec\theta\,d\theta = 9\int(\sec^3\theta - \sec\theta)\,d\theta$$

<u>REDUCTION FORMULA</u>:

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

SO

$$\int \frac{x^2}{\sqrt{9+x^2}}\,dx = 9\int\sec^3\theta\,d\theta - 9\int\sec\theta\,d\theta$$

$$= \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta+\tan\theta|$$

$$- 9\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{9}{2}\left(\frac{\sqrt{9+x^2}}{3}\right)\left(\frac{x}{3}\right) - \frac{9}{2}\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

$$= \frac{1}{2}x\sqrt{9+x^2} - \frac{9}{2}\ln\left|\sqrt{9+x^2} + x\right| + C$$

<u>QUESTION</u>: WHERE'D THE 3 GO?

5. $\displaystyle\int \frac{1}{x^2-4x+5}\,dx$

<u>NOTE</u>: NO SIMPLE SUBSTITUTION WILL WORK AND THE DENOMINATOR DOES NOT FACTOR. NEW IDEA: <u>COMPLETE THE SQUARE</u>

$$x^2 - 4x + 5 = (x^2 - 4x + \underline{\quad}) + 5 - \underline{\quad}$$

ADD THE SQUARE OF HALF THE COEFFICIENT OF X

AND SUBTRACT IT HERE

$$= (x^2 - 4x + 4) + 5 - 4$$

$$= (x-2)^2 + 1$$

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x-2)^2+1}\,dx = \int \frac{1}{u^2+1}\,du =$$

$$u = x-2$$
$$du = dx$$

$$\arctan u + C = \arctan(x-2) + C$$

6. $$\int \frac{x}{x^2-4x+8}\,dx =$$

$$x^2-4x+8 = (x^2-4x+4)+8-4 = (x-2)^2+4$$

$$\int \frac{x}{(x-2)^2+4}\,dx = \int \frac{u+2}{u^2+4}\,du = \int \left(\frac{u}{u^2+4}+\frac{2}{u^2+4}\right)du =$$

$u = x-2 \quad (x = u+2)$

$du = dx$

$$\int \frac{u}{u^2+4}\,du + 2\int \frac{1}{u^2+4}\,du = \frac{1}{2}\int \frac{1}{u^2+4}(2u\,du) + 2\int \frac{1}{4\left(\left(\frac{u}{2}\right)^2+1\right)}\,du$$

$$= \frac{1}{2}\ln|u^2+4| + \frac{1}{2}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\,du$$

$$= \frac{1}{2}\ln(u^2+4) + \frac{1}{2}\cdot 2\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\left(\frac{1}{2}\,du\right)$$

$$= \frac{1}{2}\ln(u^2+4) + \arctan\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln\left((x-2)^2+4\right) + \arctan\left(\frac{x-2}{2}\right) + C$$

$$= \frac{1}{2}\ln(x^2-4x+8) + \arctan\left(\frac{x-2}{2}\right) + C$$

7. $$\int \frac{dx}{\sqrt{5-4x-2x^2}}$$

$$5-4x-2x^2 = -2x^2-4x+5 = -2(x^2+2x \qquad)+5$$

$$= -2(x^2+2x+1)+5+2$$

ADDING 1 INSIDE THE PARENTHESES REALLY ADDS A −2.

$$5-4x-2x^2 = -2(x+1)^2+7$$

$$= 7-2(x+1)^2 = 7-(\sqrt{2}(x+1))^2$$

$$= 7\left(1-\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)^2\right)$$

$$\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx = \int \frac{1}{\sqrt{7}\sqrt{1-\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)^2}}\,dx =$$

$$\frac{1}{\sqrt{7}} \int \frac{1}{\sqrt{1-\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)^2}}\,dx = \frac{1}{\sqrt{7}}\frac{\sqrt{7}}{\sqrt{2}} \int \frac{1}{\sqrt{1-u^2}}\,du =$$

$$u = \frac{\sqrt{2}(x+1)}{\sqrt{7}}$$

$$du = \frac{\sqrt{2}}{\sqrt{7}}\,dx$$

$$\frac{1}{\sqrt{2}} \arcsin u + C = \frac{1}{\sqrt{2}} \arcsin\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right) + C$$