A FEW DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS :

RECALL : GIVEN $f \in C^\infty(M)$,

$$df : \mathcal{T}(TM) \to C^\infty(M)$$

$$df(V) = V(f), \text{ WHERE}$$

$$V(f)(p) = V_p(f)$$

E.G., LOCALLY, $dx^1, \dots, dx^n$.

df IS A $C^\infty(M)$-MODULE HOMOMORPHISM :

$$df(V_1 + V_2) = df(V_1) + df(V_2)$$

$$df(gV) = g\,(df(V)) \quad \forall g \in C^\infty(M)$$

A $C^\infty(M)$-MODULE HOMOMORPHISM

$$\theta : \mathcal{T}(TM) \to C^\infty(M)$$

IS A 1-FORM ON M.

LOCALLY,

$$\theta(V) = \theta\left(V^i \frac{\partial}{\partial x^i}\right) = V^i \theta\left(\frac{\partial}{\partial x^i}\right)$$

$$= \theta\left(\frac{\partial}{\partial x^i}\right) dx^i(V)$$

$$\theta = \theta_i \, dx^i, \text{ WHERE } \theta_i = \theta\left(\frac{\partial}{\partial x^i}\right).$$

THE SPACE OF ALL 1-FORMS ON M IS DENOTED $\Gamma(T^*M)$ AND IS A $C^\infty(M)$-MODULE :

$$(\theta_1+\theta_2)(V) = \theta_1(V)+\theta_2(V)$$

$$(g\theta)(V) = g(\theta(V))$$

NOW SUPPOSE M HAS A RIEMANNIAN METRIC g. THEN

$$V \in \Gamma(TM) \longrightarrow g(V,\cdot) : \Gamma(TM) \to C^\infty(M)$$

(" DOTTING WITH V ") IS A $C^\infty(M)$-MODULE ISOMORPHISM FROM VECTOR FIELDS TO 1-FORMS.

$$\Gamma(TM) \cong \Gamma(T^*M)$$

ON (M,g), VECTOR FIELDS AND 1-FORMS CAN BE REGARDED AS TWO DIFFERENT VIEWS OF THE SAME OBJECT.

LOCALLY,

$$V^i \frac{\partial}{\partial x^i} \longleftrightarrow (g_{ij}V^j)dx^i = V_i\, dx^i$$

OR

$$V_i\, dx^i \longleftrightarrow (g^{ij}V_j)\frac{\partial}{\partial x^i} = V^i \frac{\partial}{\partial x^i}$$

EXERCISE : PROVE THESE.

BECAUSE OF THE ISOMORPHISM $\mathcal{T}(T^*M) \cong \mathcal{T}(TM)$, EVERY 1-FORM ON M CORRESPONDS TO A UNIQUE VECTOR FIELD ON M (THE 1-FORM IS JUST "DOTTING WITH" THIS VECTOR FIELD).

E.G., $\forall f \in C^\infty(M)$, $df \in \mathcal{T}(T^*M)$ SO $\exists!$ VECTOR FIELD ON M, CALLED THE <u>GRADIENT</u> OF f AND DENOTED

$$\operatorname{grad} f \in \mathcal{T}(TM)$$

SUCH THAT

$$df(v) = g(\operatorname{grad} f, v) \quad \forall v \in \mathcal{T}(TM)$$

$$v(f) = g(\operatorname{grad} f, v) \quad \forall v \in \mathcal{T}(TM)$$

("DOTTING A VECTOR FIELD V WITH $\operatorname{grad} f$ GIVES THE DIRECTIONAL DERIVATIVE OF f IN THE DIRECTION OF V AT EACH POINT")

LOCALLY,

$$df = \frac{\partial f}{\partial x^i} dx^i \longrightarrow \operatorname{grad} f = \left(g^{ij} \frac{\partial f}{\partial x^j}\right) \frac{\partial}{\partial x^i}$$

<u>NOTE</u>: IF $x^1, \dots, x^n$ ARE STANDARD COORDINATES IN $\mathbb{R}^n$ WITH ITS STANDARD RIEMANNIAN METRIC, THEN $g^{ij} = \delta^{ij}$ SO

$$\frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n \longrightarrow \frac{\partial f}{\partial x^1} \frac{\partial}{\partial x^1} + \dots + \frac{\partial f}{\partial x^n} \frac{\partial}{\partial x^n}$$

$$\text{grad} : C^\infty(M) \longrightarrow T(TM)$$

IS OUR FIRST DIFFERENTIAL OPERATOR ON (M, g).

THE NEXT ONE IS CALLED THE (COVARIANT) DIVERGENCE

$$\text{div} : T(TM) \longrightarrow C^\infty(M).$$

THIS ONE IS A BIT TRICKIER TO MOTIVATE BECAUSE, UNLIKE THE GRADIENT, WHOSE INTRINSIC SIGNIFICANCE IS RELATED TO DERIVATIVES ($\text{grad} f$ IS SOMETHING TO DOT WITH V TO PRODUCE THE DIRECTIONAL DERIVATIVE OF f IN THE DIRECTION OF V), THE INTRINSIC SIGNIFICANCE OF div IS RELATED TO INTEGRALS (E.G., IN THE DIVERGENCE THEOREM).

OUR NEED FOR THE COVARIANT DIVERGENCE, HOWEVER, IS PRIMARILY AS A STEP TOWARD THE DEFINITION OF THE "LAPLACIAN" Δ_g ON (M, g) (RECALL THAT, ON $\mathbb{R}^n$, $\Delta = \text{div}(\text{grad})$).

THUS, WE WILL JUST FIND A COORDINATE-FREE WAY OF FORMULATING THE USUAL DIVERGENCE ON $\mathbb{R}^n$ SO THAT WE CAN APPLY IT DIRECTLY TO ANY (M, g).

<u>MOTIVATION</u> : $\mathbb{R}^n$ WITH STANDARD COORDINATES $x^1, \dots, x^n$ AND $V \in \Gamma(T\mathbb{R}^n)$. CALCULUS DEFINITION OF $\operatorname{div} V$: WRITE $V = v^i \frac{\partial}{\partial x^i}$. THEN

$$\operatorname{div} V = \frac{\partial v^1}{\partial x^1} + \dots + \frac{\partial v^n}{\partial x^n}$$

REPHRASE AS FOLLOWS : LET ∇ BE THE STANDARD (LEVI-CIVITA) CONNECTION ON $\mathbb{R}^n$.

$$\begin{aligned}\nabla_X V &= \nabla_X \left(v^i \frac{\partial}{\partial x^i} \right) = X(v^i) \frac{\partial}{\partial x^i} \\ &= \left(X^j \frac{\partial}{\partial x^j} \right)(v^i) \frac{\partial}{\partial x^i} \\ &= X^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial x^i}\end{aligned}$$

FOR V FIXED, THINK OF THIS AS A LINEAR MAP

$$X \longrightarrow \nabla_X V .$$

WE'LL COMPUTE THE TRACE (IN STANDARD COORDINATES) :

$$\begin{aligned}\partial_k \longrightarrow \nabla_{\partial_k} V &= \delta^{kj} \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial x^i} \\ &= \frac{\partial v^i}{\partial x^k} \frac{\partial}{\partial x^i}\end{aligned}$$

MATRIX : $\left(\frac{\partial v^i}{\partial x^k} \right)_{i,k = 1, \dots, n}$

TRACE : $\sum_{i=1}^n \frac{\partial v^i}{\partial x^i} = \operatorname{div} V$

THUS, A COORDINATE-FREE DEFINITION OF THE DIVERGENCE OF A VECTOR FIELD V ON $\mathbb{R}^n$ IS

$$\operatorname{div} V = \operatorname{trace} (X \to \nabla_X V)$$

AND THIS WORKS QUITE NICELY ON ANY (M, g) :

LET (M, g) BE A RIEMANNIAN n-MANIFOLD AND $V \in T'(TM)$. THE (<u>COVARIANT</u>) <u>DIVERGENCE</u> $\operatorname{div} V$ IS THE ELEMENT OF $C^\infty(M)$ DEFINED, AT EACH $p \in M$, BY

$$\operatorname{div} V(p) = \operatorname{trace} \left(x \in T_p(M) \to \nabla_x V \in T_p(M) \right)$$

NOW WE'D LIKE A FORMULA FOR COMPUTING $\operatorname{div} V$ IN LOCAL COORDINATES. AS USUAL, WE JUST COMPUTE THE TRACE IN A LOCAL COORDINATE BASIS $\{ \partial_1, \dots, \partial_n \}$.

MATRIX :

$$\begin{aligned} \partial_i \to \nabla_{\partial_i} V = \nabla_{\partial_i} (V^j \partial_j) &= V^j \nabla_{\partial_i} \partial_j + \frac{\partial V^j}{\partial x^i} \partial_j \\ &= V^j \Gamma^k_{ij} \partial_k + \frac{\partial V^k}{\partial x^i} \partial_k \\ &= \left(\frac{\partial V^k}{\partial x^i} + \Gamma^k_{ij} V^j \right) \partial_k \end{aligned}$$

$$\left(\frac{\partial V^k}{\partial x^i} + \Gamma^k_{ij} V^j \right)_{i,k = 1,\dots,n}$$

TRACE :

$$\operatorname{div} V = \frac{\partial v^i}{\partial x^i} + T^i_{ij} v^j$$

NOTE : SUM IS OVER BOTH i AND j.

FINALLY, WE DEFINE THE LAPLACIAN ON (M, g)

$$\Delta_g : C^\infty(M) \longrightarrow C^\infty(M)$$

BY

$$\Delta_g f = \operatorname{div}(\operatorname{grad} f)$$

$\forall f \in C^\infty(M)$.

WE NEED A FORMULA FOR COMPUTING THIS IN LOCAL COORDINATES SO WE BEGIN WITH

$$\begin{aligned} \Delta_g f &= \operatorname{div}\left(g^{ik} \frac{\partial f}{\partial x^k} \partial_i \right) \\ &= \frac{\partial}{\partial x^i}\left(g^{ik} \frac{\partial f}{\partial x^k} \right) + T^i_{ij}\left(g^{jk} \frac{\partial f}{\partial x^k} \right) \\ &= g^{ik} \frac{\partial^2 f}{\partial x^i \partial x^k} + \frac{\partial g^{ik}}{\partial x^i} \frac{\partial f}{\partial x^k} + T^i_{ij}\, g^{jk} \frac{\partial f}{\partial x^k} \end{aligned}$$

THIS ISN'T QUITE WHAT WE WANT, HOWEVER.

EXERCISE : $\dfrac{1}{\sqrt{\det(g)}} \dfrac{\partial}{\partial x^i}\left(g^{ik}\sqrt{\det(g)}\,\dfrac{\partial f}{\partial x^k}\right) =$

$$g^{ik}\frac{\partial^2 f}{\partial x^i \partial x^k} + \frac{\partial g^{ik}}{\partial x^i}\frac{\partial f}{\partial x^k} + \frac{1}{2\det(g)}\frac{\partial(\det(g))}{\partial x^j}\, g^{jk}\,\frac{\partial f}{\partial x^k}$$

COMPARING THIS WITH THE EXPRESSION FOR $\Delta_g f$ WE SEE THAT WE WILL OBTAIN

$$\boxed{\Delta_g f = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(g^{ik}\sqrt{\det(g)}\,\frac{\partial f}{\partial x^k}\right)}$$

IF ONLY WE CAN SHOW THAT

$$\Gamma^{i}_{ij} = \frac{1}{2\det(g)}\frac{\partial(\det(g))}{\partial x^j}$$

NOW,

$$\Gamma^{i}_{ij} = \tfrac{1}{2} g^{im}\left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right)$$

$$= \tfrac{1}{2}\left[\, g^{im}\partial_i \partial_{mj} + g^{im}\partial_j g_{mi} - g^{im}\partial_m g_{ij}\,\right]$$

$$\uparrow$$

$$g^{mi}\partial_i g_{mj} = g^{im}\partial_m g_{ij}$$

$$= \tfrac{1}{2} g^{im}\partial_j g_{mi}$$

SO WE NEED ONLY SHOW THAT

$$g^{im}\partial_j g_{mi} = \frac{1}{\det(g)}\frac{\partial(\det(g))}{\partial x^j}$$

THE PROOF IS " JUST A CALCULATION ", AS THEY SAY, BUT IN DIMENSION n ITS A PRETTY ANNOYING ONE.

FOR $n = 2$,

$$\det(g) = g_{11} g_{22} - (g_{12})^2$$

SO IT'S SIMPLE AND I'LL LEAVE IT FOR YOU.

<u>EXERCISE</u> : COMPUTE $\dfrac{1}{g_{11} g_{22} - (g_{12})^2} \dfrac{\partial}{\partial x^j} (g_{11} g_{22} - (g_{12})^2)$, $j = 1, 2$, AND SHOW THAT IT IS $g^{im} \partial_j g_{mi}$.

SINCE WE WILL ONLY USE THE RESULT FOR RIEMANNIAN SURFACES WE CAN BE HAPPY WITH THIS MUCH.

<u>NOTE</u> : FOR $M = \mathbb{R}^n$ WITH ITS STANDARD METRIC, DENOTED g_0 FROM NOW ON, $\det(g) = 1$ AND $g^{ik} = \delta^{ik}$ SO THE FORMULA IN THE BOX REDUCES TO

$$\Delta_{g_0} f = \sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2}$$

(THE "USUAL" LAPLACIAN IN STANDARD COORDINATES)

NOW WE CAN TAKE THE FIRST STEP TOWARD

THE UNIFORMIZATION THEOREM : LET (M, g) BE A COMPACT, 2-DIMENSIONAL, RIEMANNIAN MANIFOLD. THEN THERE IS A RIEMANNIAN METRIC

$$\tilde{g} = e^{2\mu} g \qquad (\mu \in C^\infty(M))$$

CONFORMAL TO g WHICH HAS CONSTANT GAUSSIAN CURVATURE $K_{\tilde{g}}$.

THE IDEA HERE IS TO WRITE AN EXPLICIT FORMULA RELATING THE TWO GAUSSIAN CURVATURES $K_{\tilde{g}}$ AND K_g AND TRY TO CHOOSE μ IN SUCH A WAY AS TO MAKE $K_{\tilde{g}}$ COME OUT CONSTANT.

MORE EXPLICITLY, WE WILL SHOW THAT

$$\boxed{K_{\tilde{g}} = (K_g - \Delta_g \mu) e^{-2\mu}}$$

SO THAT, FOR EXAMPLE, FINDING A μ FOR WHICH $K_{\tilde{g}} \equiv 0$ BOILS DOWN TO PROVING THE EXISTENCE OF A SOLUTION TO THE PARTIAL DIFFERENTIAL EQUATION

$$\Delta_g \mu = K_g$$

(g AND K_g ARE KNOWN).

PROVING THIS RELATIONSHIP BETWEEN K_g AND $K_{\tilde{g}}$ DIRECTLY BY BRUTE FORCE IS QUITE A MESS, BUT CAN BE SIMPLIFIED CONSIDERABLY IF WE USE THE EXISTENCE OF LOCAL <u>ISOTHERMAL</u> <u>COORDINATES</u>. WE HAVE MENTIONED THIS RESULT BEFORE AS WELL AS THE FACT THAT IT IS QUITE NONTRIVIAL. NEVERTHELESS, WE WILL USE IT HERE.

THUS, FOR THE RIEMANNIAN SURFACE (M, g) WE SELECT LOCAL COORDINATES x^1, x^2 FOR WHICH

$$g = e^{2\nu}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) = e^{2\nu} g_0$$

$$g_{11} = g_{22} = e^{2\nu}, \quad g_{12} = g_{21} = 0$$

WE FIRST PROVE THE SPECIAL CASE OF OUR FORMULA RELATING K_g AND K_{g_0}. SINCE $K_{g_0} \equiv 0$ THIS IS

$$K_g = (-\Delta_{g_0} \nu)\, e^{-2\nu}$$

SHOWING THIS IS ROUTINE SINCE ISOTHERMAL COORDINATES ARE, IN PARTICULAR, ORTHOGONAL, SO WE CAN COMPUTE K_g USING

$$K_g = -\frac{1}{2\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial x^1}\left(\frac{\frac{\partial g_{22}}{\partial x^1}}{\sqrt{g_{11}g_{22}}}\right)+\frac{\partial}{\partial x^2}\left(\frac{\frac{\partial g_{11}}{\partial x^2}}{\sqrt{g_{11}g_{22}}}\right)\right]$$

WITH $g_{11} = g_{22} = e^{2\nu}$. THIS IS REALLY BORING, HOWEVER, SO I WON'T DO IT. IF YOU DO, YOU'LL GET

$$K_g = -\left(\frac{\partial^2\nu}{(\partial x^1)^2}+\frac{\partial^2\nu}{(\partial x^2)^2}\right)e^{-2\nu}$$

$$= (-\Delta_{g_0}\nu)e^{-2\nu},$$

AS REQUIRED.

NOW RETURN TO THE GENERAL CASE : $\tilde{g} = e^{2\mu}g$

IN ISOTHERMAL COORDINATES, $g = e^{2\nu}g_0$ SO

$$\tilde{g} = e^{2(\mu+\nu)}g_0$$

ACCORDING TO THE SPECIAL CASE WE HAVE JUST PROVED,

$$K_{\tilde{g}} = (-\Delta_{g_0}(\mu+\nu))e^{-2(\mu+\nu)}$$

$$= (-\Delta_{g_0}\mu - \Delta_{g_0}\nu)\,e^{-2\mu}e^{-2\nu}$$

$$= ((-\Delta_{g_0}\mu)e^{-2\nu} + (-\Delta_{g_0}\nu)e^{-2\nu})\,e^{-2\mu}$$

$$= ((-\Delta_{g_0}\mu)e^{-2\nu} + K_g)\,e^{-2\mu}$$

$$\mathcal{K}_{\tilde{g}} = (\mathcal{K}_g - (\Delta_{g_0}\mu)e^{-2\nu})e^{-2\mu}$$

SO ALL THAT'S LEFT IS TO SHOW THAT, FOR

$$g_{11} = g_{22} = e^{2\nu} \quad , \quad g_{12} = g_{21} = 0 ,$$

$$\Delta_g \mu = (\Delta_{g_0}\mu)e^{-2\nu}$$

BUT

$$\Delta_g \mu = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x^i}\left(g^{ik} \sqrt{\det(g)} \frac{\partial \mu}{\partial x^k} \right)$$

$$\det(g) = e^{4\nu}$$

$$g^{11} = \frac{1}{\det(g)} g_{22} = e^{-2\nu}$$

$$g^{22} = \frac{1}{\det(g)} g_{11} = e^{-2\nu}$$

$$g^{12} = g^{21} = 0$$

SO

$$\Delta_g \mu = e^{-2\nu}\left[\frac{\partial}{\partial x^1}\left(e^{-2\nu} e^{2\nu} \frac{\partial \mu}{\partial x^1} \right) + \frac{\partial}{\partial x^2}\left(e^{-2\nu} e^{2\nu} \frac{\partial \mu}{\partial x^2} \right) \right]$$

$$= e^{-2\nu}\left[\frac{\partial^2 \mu}{(\partial x^1)^2} + \frac{\partial^2 \mu}{(\partial x^2)^2} \right]$$

$$= (\Delta_{g_0}\mu)e^{-2\nu}$$

AS REQUIRED.

THE BOTTOM LINE THEN IS THAT WE HAVE PROVED

$$\tilde{g} = e^{2\mu} g \quad \Rightarrow \quad K_{\tilde{g}} = (K_g - \Delta_g \mu) e^{-2\mu} .$$

SO, TO PROVE THE UNIFORMIZATION THEOREM, HERE'S WHAT YOU DO :

GIVEN THE COMPACT, 2-DIMENSIONAL, RIEMANNIAN MANIFOLD (M, g),

1. IF $\frac{1}{2\pi} \int_M K_g = \chi(M) = 0$, PROVE THE EXISTENCE OF A SMOOTH SOLUTION TO

$$\boxed{\Delta_g \mu = K_g} \qquad \boxed{K_{\tilde{g}} = 0}$$

2. IF $\frac{1}{2\pi} \int_M K_g = \chi(M) < 0$, PROVE THE EXISTENCE OF A SMOOTH SOLUTION TO

$$\boxed{\Delta_g \mu - e^{2\mu} = K_g} \qquad \boxed{K_{\tilde{g}} = -1}$$

3. IF $\frac{1}{2\pi} \int_M K_g = \chi(M) > 0$, PROVE THE EXISTENCE OF A SMOOTH SOLUTION TO

$$\boxed{\Delta_g \mu + e^{2\mu} = K_g} \qquad \boxed{K_{\tilde{g}} = 1}$$