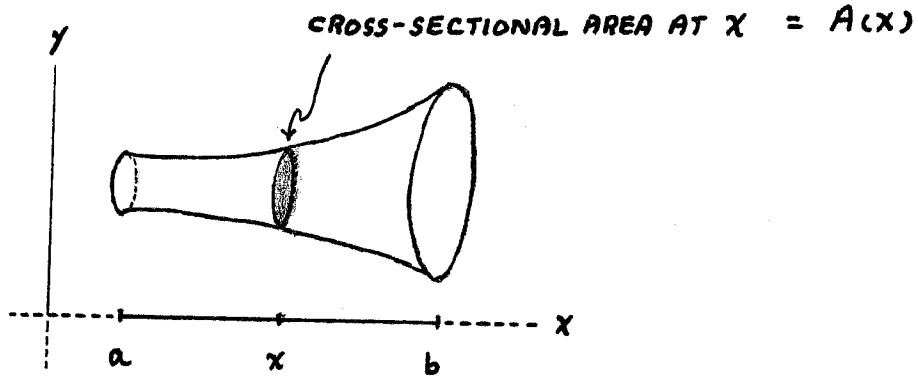


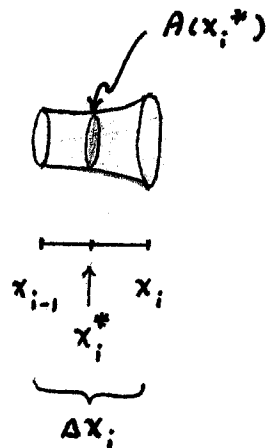
VOLUMES I

THE VOLUME OF A SOLID CAN BE COMPUTED BY "INTEGRATING ITS CROSS-SECTIONAL AREA"



$$\text{VOLUME} = \int_a^b A(x) dx$$

HERE'S THE REASON : APPROXIMATE THE VOLUME BY CARVING IT INTO THIN CYLINDRICAL SLABS



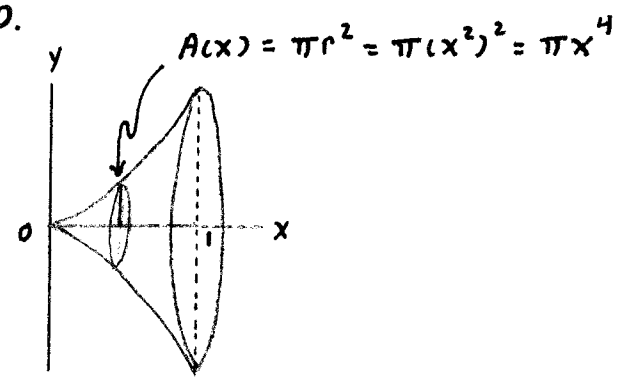
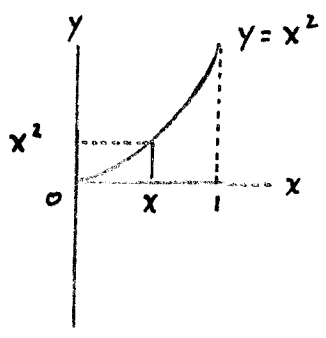
$$\text{VOLUME} \approx A(x_i^*) \Delta x_i$$

ADD $\sum_{i=1}^n A(x_i^*) \Delta x_i$. APPROXIMATION IMPROVES AS SLABS GET THINNER SO

$$\text{VOLUME} = \lim_{\Delta x_{\max} \rightarrow 0} \sum_{i=1}^n A(x_i^*) \Delta x_i = \int_a^b A(x) dx$$

EXAMPLES :

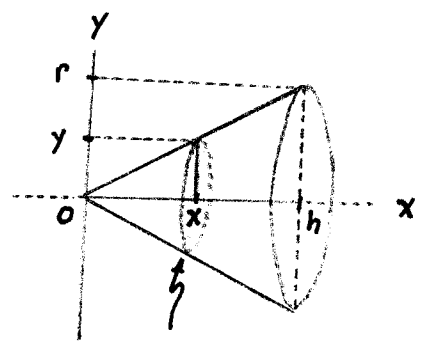
1. THE REGION UNDER THE GRAPH OF $y = f(x) = x^2$ AND ABOVE THE INTERVAL $[0, 1]$ IS REVOLVED AROUND THE X-AXIS. COMPUTE THE VOLUME OF THE RESULTING SOLID.



$$V = \int_0^1 A(x) dx = \int_0^1 \pi x^4 dx$$

$$= \pi \left(\frac{1}{5} x^5 \Big|_0^1 \right) = \frac{\pi}{5}$$

2. FIND THE VOLUME OF A CONE OF HEIGHT h AND RADIUS r .



SIMILAR TRIANGLES :

$$\frac{y}{x} = \frac{r}{h}$$

$$y = \frac{r}{h} x$$

$$A(x) = \pi r^2$$

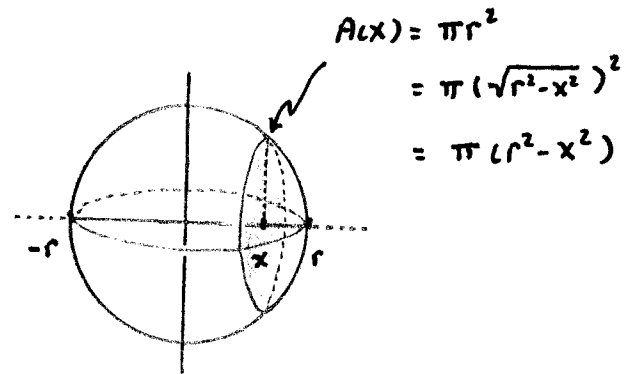
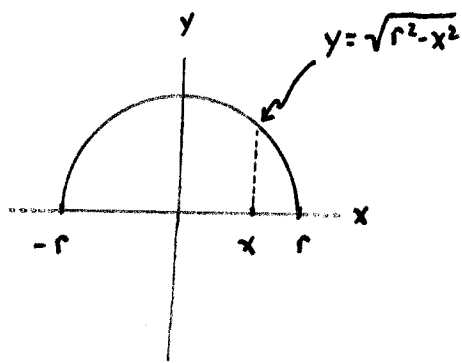
$$= \pi y^2 = \pi \left(\frac{r}{h} x \right)^2$$

$$= \frac{\pi r^2}{h^2} x^2$$

$$V = \int_0^h A(x) dx = \frac{\pi r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \frac{1}{3} x^3 \Big|_0^h = \frac{1}{3} \pi r^2 h$$

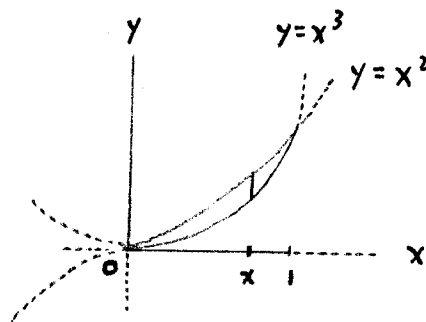
3. FIND THE VOLUME OF A SPHERE OF RADIUS r .

THINK OF THE SPHERE AS THE SOLID OBTAINED BY REVOLVING THE REGION UNDER $y = f(x) = \sqrt{r^2 - x^2}$ AND ABOVE THE INTERVAL $[-r, r]$ ABOUT THE x -AXIS.



$$\begin{aligned}
 V &= \int_{-r}^r A(x) dx = \int_{-r}^r \pi(r^2 - x^2) dx = \pi r^2 x \Big|_{-r}^r - \frac{\pi}{3} x^3 \Big|_{-r}^r \\
 &= \pi r^2 (r - (-r)) - \frac{\pi}{3} (r^3 - (-r)^3) = 2\pi r^3 - \frac{2\pi}{3} r^3 \\
 &= \frac{4}{3} \pi r^3
 \end{aligned}$$

4. REVOLVE THE REGION BETWEEN THE GRAPHS OF $y = x^2$ AND $y = x^3$ ABOUT THE x -AXIS. COMPUTE THE VOLUME OF THE RESULTING SOLID.



INTERSECTIONS:

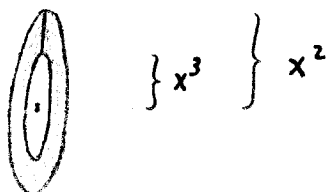
$$x^3 = x^2$$

$$x^3 - x^2 = 0$$

$$x^2(x-1) = 0$$

$$x = 0, 1$$

THIS TIME THE CROSS-SECTION AT x ISN'T A CIRCLE, BUT A WASHER (ANNULUS).



$$A(x) = (\text{AREA OF LARGE CIRCLE}) - (\text{AREA OF SMALL CIRCLE})$$

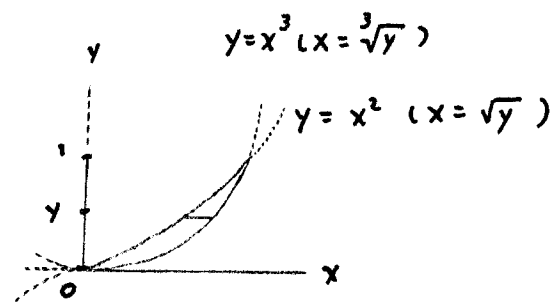
$$= \pi (x^2)^2 - \pi (x^3)^2$$

$$= \pi x^4 - \pi x^6$$

$$V = \int_0^1 A(x) dx = \pi \int_0^1 (x^4 - x^6) dx = \pi \left[\frac{1}{5} x^5 \Big|_0^1 - \frac{1}{7} x^7 \Big|_0^1 \right]$$

$$= \pi \left[\frac{1}{5} - \frac{1}{7} \right] = \frac{2\pi}{35}$$

5. THIS TIME WE'LL REVOLVE THE REGION IN EXAMPLE #4 ABOUT THE y -AXIS TO PRODUCE A SOLID AND COMPUTE ITS VOLUME.



JUST REVERSE THE ROLES OF THE VARIABLES. FOR EACH y BETWEEN 0 AND 1 THE CROSS-SECTION IS AN ANNULUS



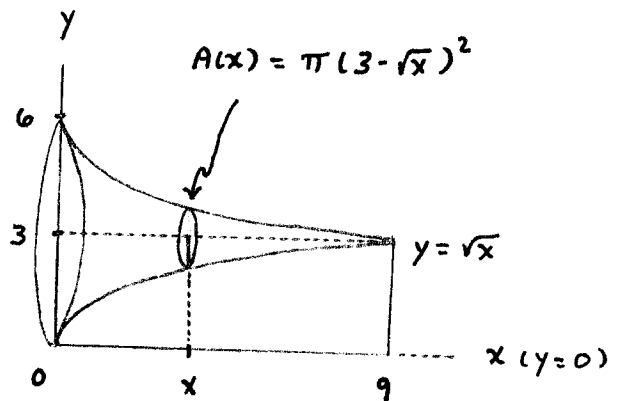
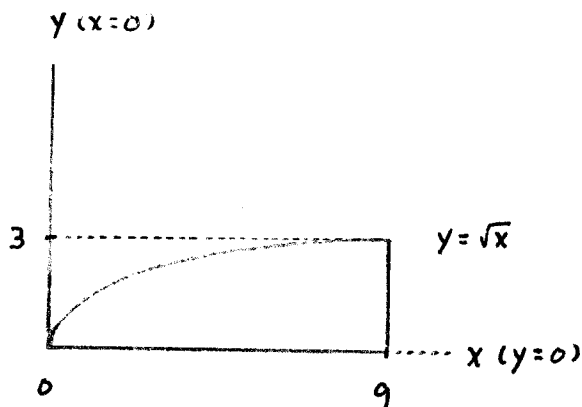
$$\sqrt{y}$$

$$\sqrt[3]{y}$$

$$A(y) = \pi (\sqrt[3]{y})^2 - \pi (\sqrt{y})^2 = \pi y^{\frac{2}{3}} - \pi y$$

$$\begin{aligned} V &= \int_0^1 A(y) dy = \pi \int_0^1 (y^{\frac{2}{3}} - y) dy = \pi \left[\frac{3}{5} y^{\frac{5}{3}} \Big|_0^1 - \frac{1}{2} y^2 \Big|_0^1 \right] \\ &= \pi \left[\frac{3}{5} - \frac{1}{2} \right] = \frac{\pi}{10} \end{aligned}$$

6. FIND THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THE REGION ENCLOSED BY $y = \sqrt{x}$, $y = 3$ AND $x = 0$ ABOUT THE LINE $y = 3$.



$$\begin{aligned} V &= \int_0^9 A(x) dx = \pi \int_0^9 (3 - \sqrt{x})^2 dx = \pi \int_0^9 (9 - 6\sqrt{x} + x) dx \\ &= \pi \left[9x \Big|_0^9 - 4x^{\frac{3}{2}} \Big|_0^9 + \frac{1}{2} x^2 \Big|_0^9 \right] = \frac{27}{2} \end{aligned}$$