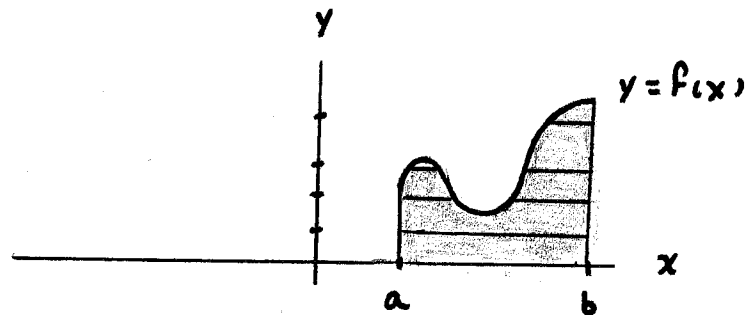


VOLUMES II

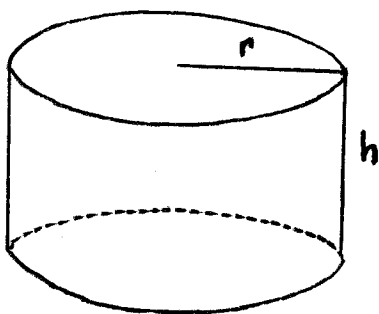
HERE IS ANOTHER VOLUME PROBLEM :

FIND THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THE REGION



ABOUT THE y -AXIS. INTEGRATING CROSS-SECTIONAL AREAS WOULD BE PRETTY ANNOYING THIS TIME SINCE THE CROSS-SECTIONS ARE OF 4 DIFFERENT TYPES.

TO DESCRIBE AN ALTERNATIVE, RECALL THE FOLLOWING :



$$\text{SURFACE AREA} = 2\pi r h$$

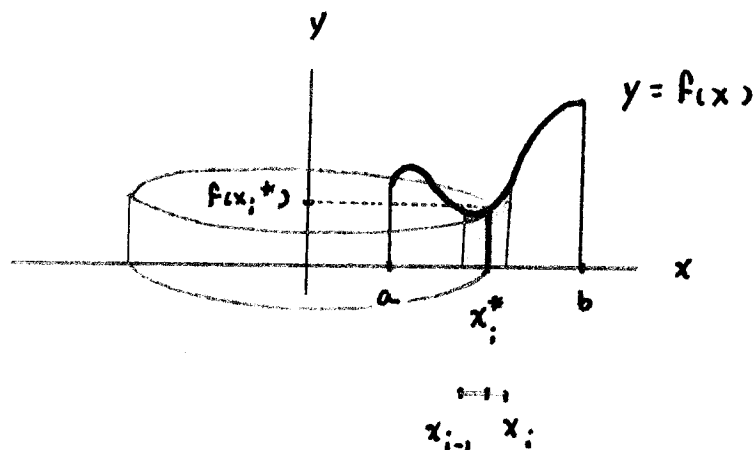
(SLICE AND FLATTEN)

NOW "THICKEN" THE CYLINDER BY Δr .

$$\text{VOLUME} = 2\pi r h \Delta r$$

(SLICE AND FLATTEN)

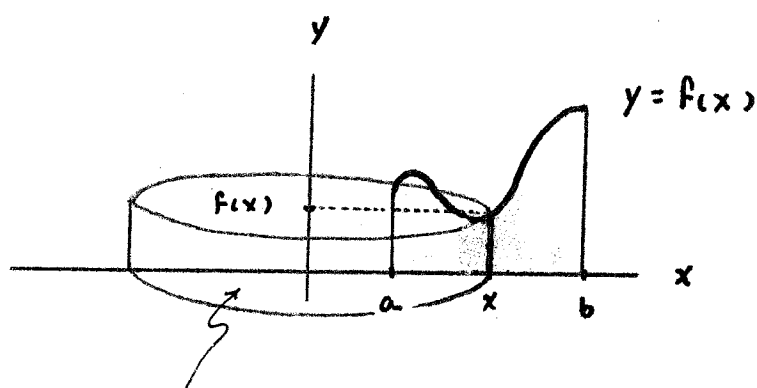
NOW RETURN TO THE SOLID DESCRIBED ABOVE :



$$\text{VOLUME} \approx \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x_i$$

$$\begin{aligned} \text{VOLUME} &= \lim_{\Delta x_{\max} \rightarrow 0} \sum_{i=1}^n 2\pi x_i^* f(x_i^*) \Delta x_i \\ &= \int_a^b 2\pi x f(x) dx \end{aligned}$$

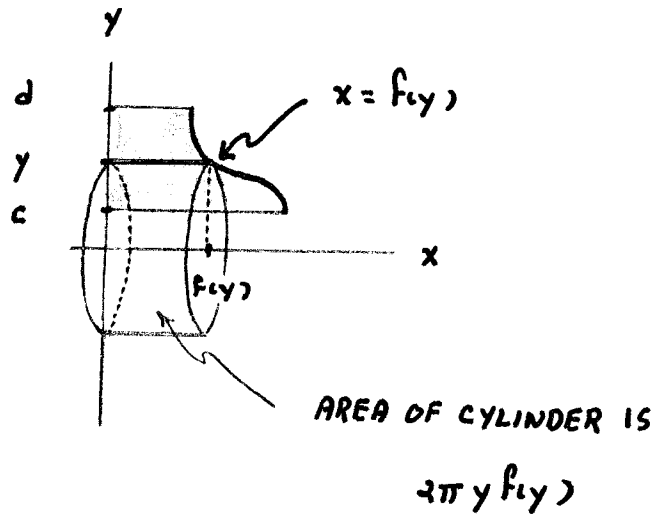
THE BEST WAY TO REMEMBER AND USE THIS FORMULA (AND ITS VARIANTS) IS AS FOLLOWS :



AREA OF CYLINDER IS $2\pi x f(x)$

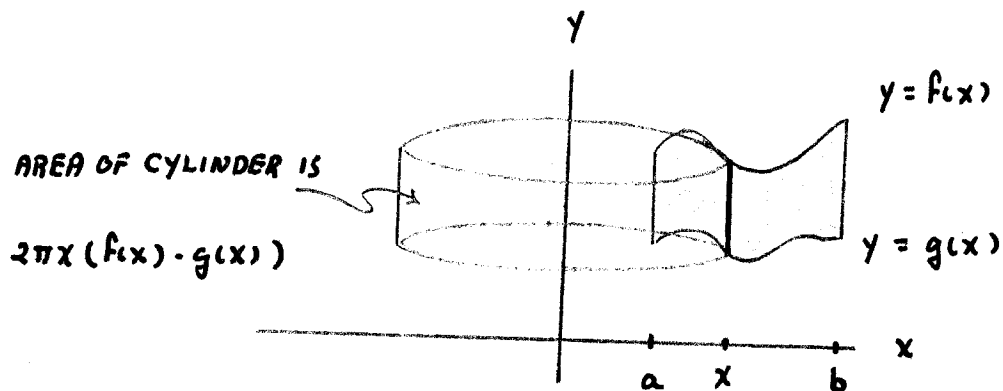
INTEGRATE THIS OVER $[a, b]$

SAME FOR REVOLVING ABOUT X-AXIS :



$$\text{VOLUME} = \int_c^d 2\pi y f(y) dy$$

SAME FOR THE REGION BETWEEN TWO GRAPHS, E.G.,

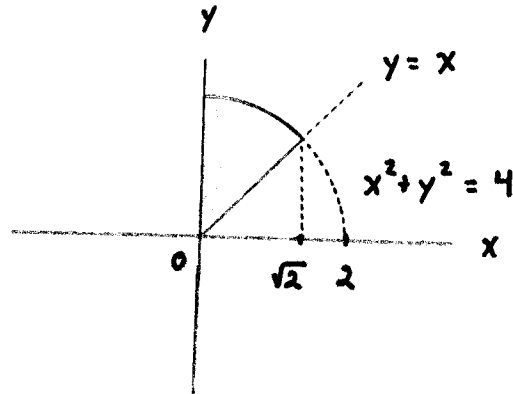


$$\text{VOLUME} = \int_a^b 2\pi x (f(x) - g(x)) dx$$

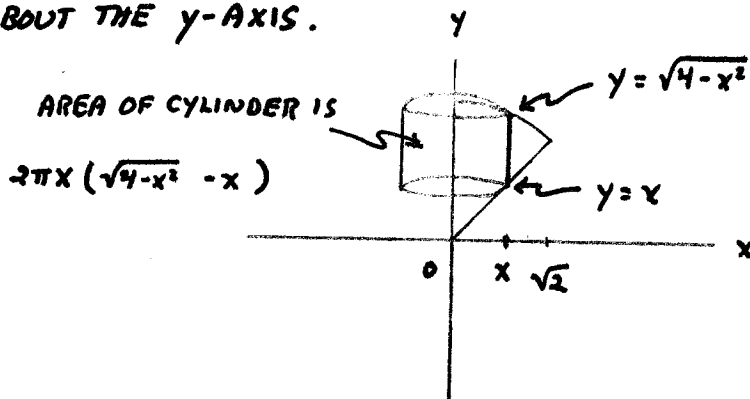
AND SIMILARLY FOR REVOLVING ABOUT THE X-AXIS.

EXAMPLES :

1. CONSIDER THE REGION IN THE FIRST QUADRANT BOUNDED BY THE GRAPHS OF $x^2 + y^2 = 4$ AND $y = x$.



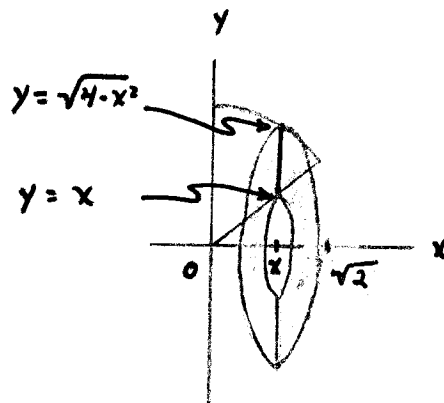
FIND THE VOLUME OF THE SOLID OBTAINED BY REVOLVING THIS REGION ABOUT THE y -AXIS.



$$\begin{aligned}
 \text{VOLUME} &= \int_0^{\sqrt{2}} 2\pi x (\sqrt{4-x^2} - x) dx \\
 &= 2\pi \int_0^{\sqrt{2}} x (4-x^2)^{\frac{1}{2}} dx - 2\pi \int_0^{\sqrt{2}} x^2 dx \\
 &\quad u = 4-x^2 \\
 &\quad du = -2x dx \\
 &\quad x=0 \Rightarrow u=4 \\
 &\quad x=\sqrt{2} \Rightarrow u=2
 \end{aligned}$$

$$\begin{aligned}
 \text{VOLUME} &= -\pi \int_0^{\sqrt{2}} (4-x^2)^{\frac{1}{2}} (-2x dx) - 2\pi \left[\frac{1}{3} x^3 \right]_0^{\sqrt{2}} \\
 &= -\pi \int_4^2 u^{\frac{1}{2}} du - 2\pi \left(\frac{2\sqrt{2}}{3} - 0 \right) \\
 &= -\pi \left[\frac{2}{3} u^{\frac{3}{2}} \right]_4^2 - \frac{4\sqrt{2}\pi}{3} \\
 &= -\frac{2\pi}{3} [2\sqrt{2} - 8] - \frac{4\sqrt{2}\pi}{3} = -\frac{2\pi}{3} [2\sqrt{2} - 8 + 2\sqrt{2}] \\
 &= -\frac{2\pi}{3} [4\sqrt{2} - 8] = \frac{8\pi}{3} [2 - \sqrt{2}]
 \end{aligned}$$

2. REVOLVE THE SAME REGION AS IN EXAMPLE 1 ABOUT THE X-AXIS AND FIND THE VOLUME OF THE RESULTING SOLID.



NOTICE THAT IN THIS CASE IT IS MORE SENSIBLE TO GO BACK TO INTEGRATING CROSS-SECTIONAL AREAS

$$\begin{aligned}
 \text{VOLUME} &= \int_0^{\sqrt{2}} A(x) dx = \int_0^{\sqrt{2}} [\pi (\sqrt{4-x^2})^2 - \pi x^2] dx \\
 &= \pi \int_0^{\sqrt{2}} [4 - 2x^2] dx = \pi \left[4x \Big|_0^{\sqrt{2}} - \frac{2}{3} x^3 \Big|_0^{\sqrt{2}} \right] \\
 &= \pi \left[4\sqrt{2} - \frac{2}{3} (2\sqrt{2}) \right] = 4\sqrt{2}\pi \left[1 - \frac{1}{3} \right] = \frac{8\sqrt{2}\pi}{3}
 \end{aligned}$$