WORK AND ENERGY

MOST PHYSICAL PHENOMENA ARE MUCH TOO COMPLICATED TO BE ANALYZED IN DETAIL. THINK OF THE ELECTROMAGNETIC INTERACTIONS BETWEEN THE ELECTRONS IN THE MOLECULES AT THE SURFACE OF YOUR HANDS AND THOSE IN THE METAL OF YOUR STALLED CAR AS YOU TRY TO PUSH IT DOWN THE ROAD.

FORTUNATELY, PHYSICS HAS DEVISED CONCEPTS AND PRINCIPLES THAT GENERALLY MAKE IT UNNECESSARY TO UNDERSTAND ALL OF THESE DETAILS IN ORDER TO PREDICT "WHAT HAPPENS". "WORK" AND "ENERGY" ARE TWO SUCH CONCEPTS AND THE "CONSERVATION OF ENERGY" IS SUCH A PRINCIPLE.

THESE ALSO PROVIDE FINE OPPORTUNITIES TO SEE HOW SUCH THINGS AS LIMITS OF RIEMANN SUMS (I.E., INTEGRALS) AND THE CHAIN RULE COME UP "IN LIFE" (SO TO SPEAK).

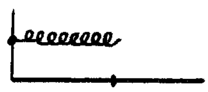
PUSH ON YOUR STALLED CAR AND MOVE IT DOWN THE STREET AND YOU DO "WORK" ON THE CAR. BUT IF THE BREAK IS ON AND YOUR CAR DOES NOT MOVE, THEN (ALTHOUGH YOU MAY WEAR YOURSELF OUT) YOU DO NO "WORK". WE NOW MAKE THIS MORE PRECISE.

IF A CONSTANT FORCE OF MAGNITUDE F IS APPLIED IN THE DIRECTION OF MOTION OF AN OBJECT AND IF, AS A RESULT, THE OBJECT MOVES A DISTANCE d, THEN THE WORK W DONE ON THE OBJECT IS, BY DEFINITION,

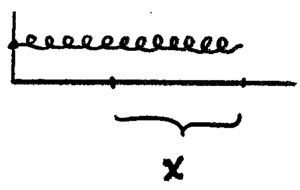
$$W = Fd .$$

UNITS :	DISTANCE	FORCE	WORK
	METER	NEWTON	JOULE (NEWTON-METER)
	CENTIMETER	DYNE	ERG (DYNE-CENTIMETER)
	FOOT	POUND	FOOT-POUND

USUALLY, FORCES APPLIED TO OBJECTS ARE NOT CONSTANT, E.G., STRETCHING A SPRING (THE MORE IT IS STRETCHED, THE GREATER THE FORCE REQUIRED TO CONTINUE STRETCHING IT).



EQUILIBRIUM



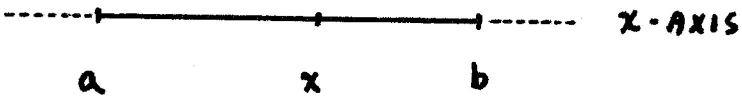
STRETCHED

HOOKE'S LAW : ANY SPRING HAS A SPRING CONSTANT k SUCH THAT, WHEN STRETCHED (OR COMPRESSED) x UNITS BEYOND ITS NATURAL LENGTH, THE SPRING PULLS BACK WITH A FORCE OF MAGNITUDE

$$F = F(x) = kx$$

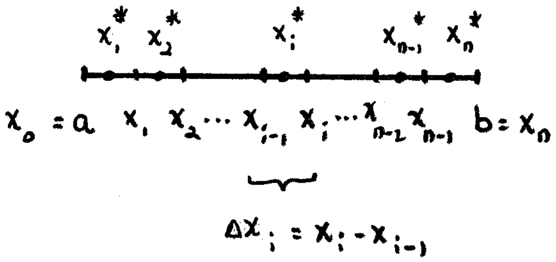
THIS IS THE FORCE THAT MUST BE APPLIED TO STRETCH THE SPRING TO x.

HOW DO YOU DEFINE / COMPUTE THE WORK DONE BY A VARIABLE FORCE (ALONG A STRAIGHT LINE, WHICH WE MAKE INTO THE x-AXIS) ?



$F(x)$ = MAGNITUDE OF APPLIED FORCE AT x

<u>STANDARD</u> <u>OPERATING</u> <u>PROCEDURE</u> : OVER A SMALL SUBINTERVAL THE FORCE IS APPROXIMATELY CONSTANT SO WE CAN APPROXIMATE THE WORK OVER THIS SUBINTERVAL WITH " $W = Fd$ ". USE LOTS OF SMALLER AND SMALLER SUBINTERVALS AND TAKE THE LIMIT.



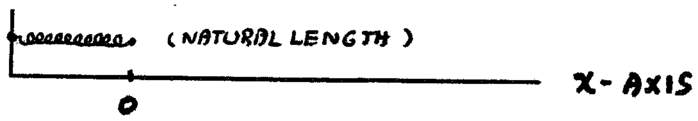
$$W_i \approx F(x_i^*)\,\Delta x_i$$

$$W \approx \sum_{i=1}^{n} W_i \qquad \sum_{i=1}^{n} F(x_i^*)\,\Delta x_i$$

$$W = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} F(x_i^*)\,\Delta x_i$$

$$W = \int_a^b F(x)\,dx$$

EXAMPLE : A SPRING EXERTS A FORCE OF 5 N WHEN STRETCHED 1 m BEYOND ITS NATURAL LENGTH. FIND THE SPRING CONSTANT AND THEN COMPUTE THE WORK DONE TO STRETCH THE SPRING 2 m BEYOND ITS NATURAL LENGTH.



$$F(x) = kx$$

$$5 = F(1) = k \cdot 1 \Rightarrow k = 5$$

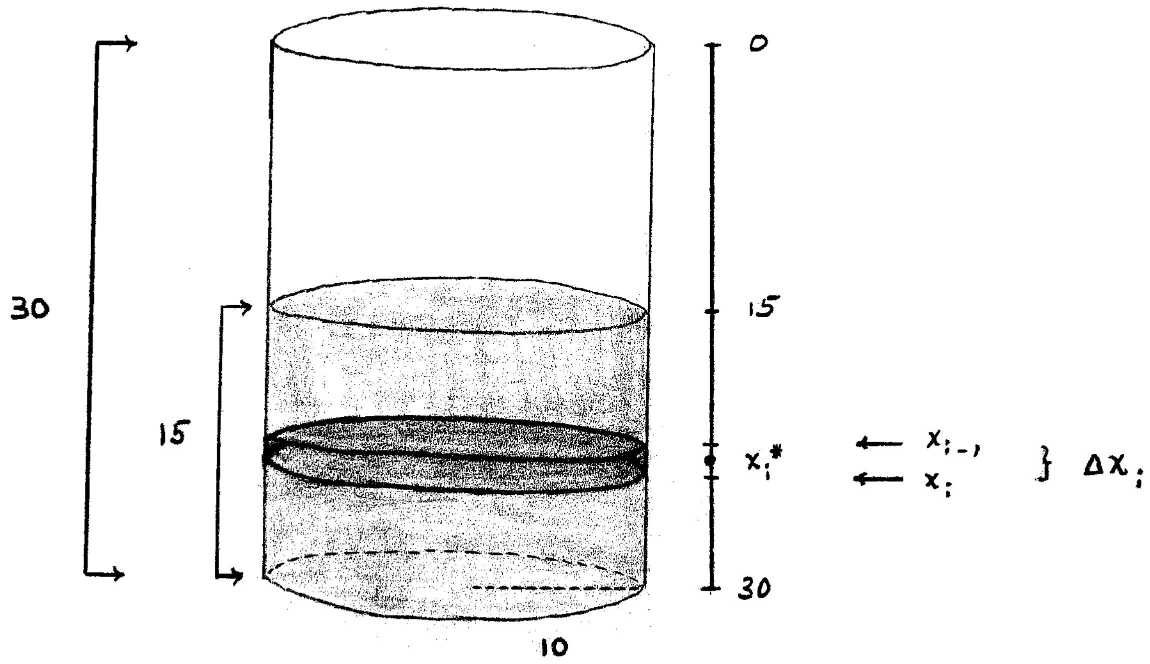
$$F(x) = 5x$$

$$W = \int_0^2 F(x)\,dx = \int_0^2 5x\,dx = \frac{5}{2}x\Big|_0^2 = 10 \text{ J}$$

USUALLY THINGS AREN'T SO SIMPLE (NO "HOOKE'S LAW") AND ONE MUST FIND OUT WHAT INTEGRAL TO EVALUATE BY ACTUALLY DOING THE RIEMANN SUM APPROXIMATION.

EXAMPLES :

1. A CYLINDRICAL TANK OF RADIUS 10 FT AND HEIGHT 30 FT IS FILLED WITH WATER TO A DEPTH OF 15 FT. HOW MUCH WORK IS REQUIRED TO PUMP ALL OF THE WATER OUT OF THE TOP OF THE TANK ?



THE FORCE EXERTED MUST OVERCOME THE WEIGHT OF THE WATER

WEIGHT DENSITY OF WATER : 62.4 lb/ft^3 , OR
9810 N/m^3

EACH "LEVEL" MUST BE LIFTED A DIFFERENT DISTANCE

W_i = WORK REQUIRED TO LIFT THE WATER BETWEEN X_{i-1} AND X_i OUT OF THE TANK

$\approx$ (WEIGHT OF THE WATER BETWEEN X_{i-1} AND X_i)(X_i^*) =

$(62.4)(\text{VOLUME})(X_i^*) =$

$(62.4)(\pi(\text{RADIUS AT } X_i^*)^2 \Delta X_i)(X_i^*) =$

$(62.4)(\pi(10)^2 \Delta X_i)(X_i^*) =$

$6240\pi X_i^* \Delta X_i$

$W_i \approx 6240\pi X_i^* \Delta X_i$

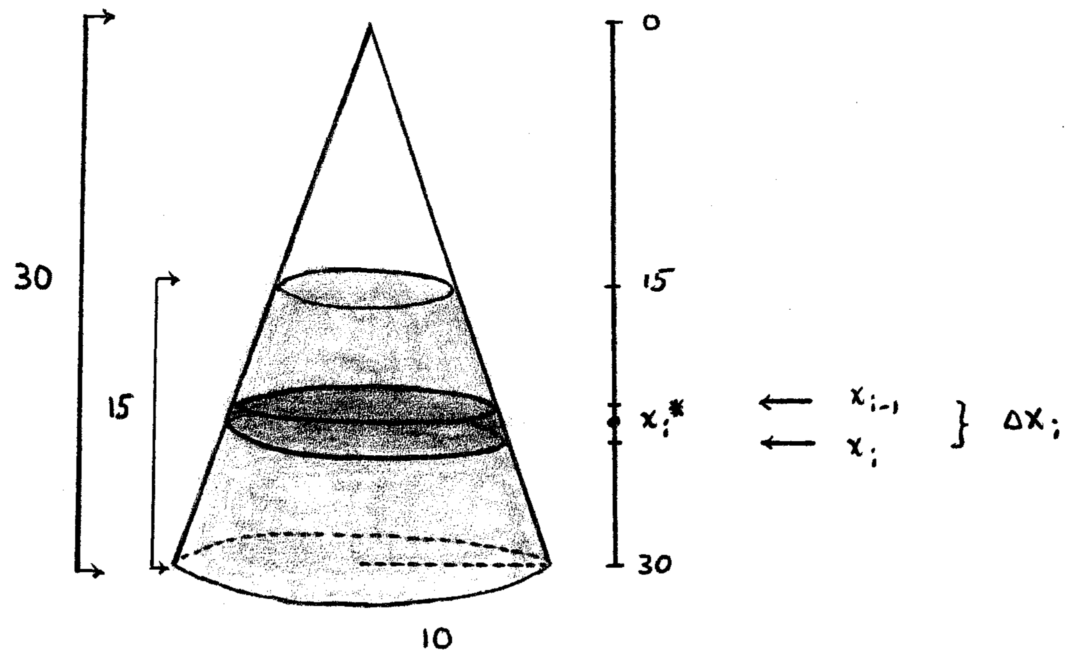
$$W \approx \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 6240\pi x_i^* \Delta x_i$$

$$W = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} 6240\pi x_i^* \Delta x_i$$

$$= \int_{15}^{30} 6240\pi x \, dx$$

$$= 3120\pi x^2 \Big|_{15}^{30} = 2{,}106{,}000\pi \text{ FT-LB}$$

2. A CONICAL TANK OF RADIUS 10 FT AND HEIGHT 30 FT IS FILLED WITH WATER TO A DEPTH OF 15 FT. HOW MUCH WORK IS REQUIRED TO PUMP ALL OF THE WATER OUT OF THE TOP OF THE TANK?



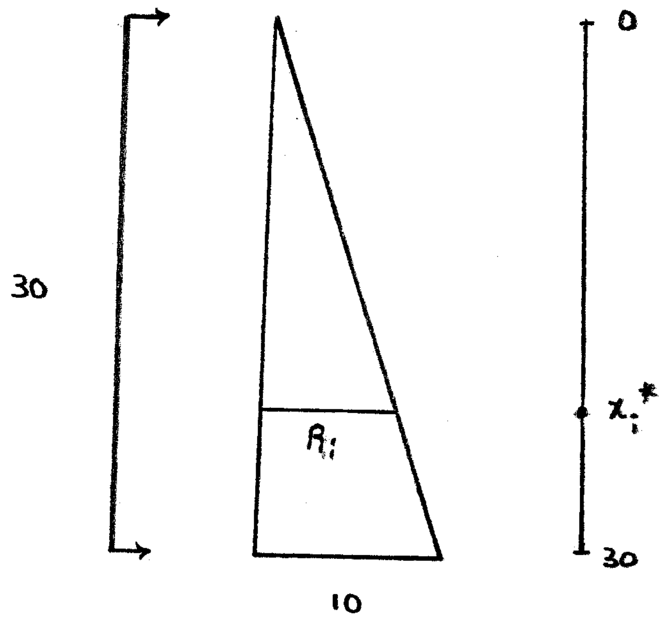
W_i = WORK REQUIRED TO LIFT THE WATER BETWEEN X_{i-1} AND X_i OUT OF THE TANK

$\approx$ (WEIGHT OF THE WATER BETWEEN X_{i-1} AND X_i) (X_i^*) =

$(62.4)(\text{VOLUME})(X_i^*)$

$\approx (62.4)(\text{AREA AT } X_i^*)(\Delta X_i)(X_i^*)$ =

$(62.4)\pi(\text{RADIUS AT } X_i^*)^2(\Delta X_i)(X_i^*)$



SIMILAR TRIANGLES:

$$\frac{X_i^*}{R_i} = \frac{30}{10}$$

$$R_i = \frac{1}{3}X_i^*$$

$$W_i \approx (62.4)\pi\left(\frac{1}{3}X_i^*\right)^2(\Delta X_i)(X_i^*) = \frac{62.4\pi}{9}(X_i^*)^3\Delta X_i$$

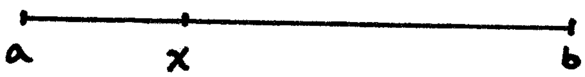
$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \frac{62.4\pi}{9} (x_i^*)^3 \Delta x_i$$

$$W = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \frac{62.4\pi}{9} (x_i^*)^3 \Delta x_i$$

$$= \int_{15}^{30} \frac{62.4\pi}{9} x^3 dx = \frac{62.4\pi}{36} x^4 \Big|_{15}^{30}$$

$$= 1{,}316{,}250\pi \text{ FT-LB}$$

NOW WE TURN TO ANOTHER WAY OF LOOKING AT WORK THAT GIVES SOME INTUITION ABOUT WHAT IT MEANS.



OBJECT MOVES FROM $x = a$ TO $x = b$ UNDER THE INFLUENCE OF A VARIABLE FORCE $F(x)$.

SUPPOSE t IS TIME AND

$$x = x(t), \quad t_0 \le t \le t_1$$

$$a = x(t_0)$$

$$b = x(t_1)$$

$$W = \int_a^b F(x)\,dx$$

NOW NOTICE THAT

$$\int_{t_0}^{t_1} F(x(t))\,x'(t)\,dt \qquad = \int_a^b F(u)\,du \;=\; W$$

$$u = x(t)$$
$$du = x'(t)\,dt$$
$$t = t_0 \Rightarrow u = x(t_0) = a$$
$$t = t_1 \Rightarrow u = x(t_1) = b$$

SO

$$W = \int_{t_0}^{t_1} F(x(t))\,x'(t)\,dt$$

NOW USE NEWTON'S SECOND LAW $(F = mA)$:

$$F(x(t)) = m\,x''(t)$$

$$\begin{aligned} W &= \int_{t_0}^{t_1} m\,x''(t)\,x'(t)\,dt \\ &= \frac{1}{2} m \int_{t_0}^{t_1} (x'(t)\,x'(t))'\,dt \\ &= \frac{1}{2} m\, x'(t)\,x'(t) \Big|_{t_0}^{t_1} \\ &= \frac{1}{2} m\,(v(t))^2 \Big|_{t_0}^{t_1} \\ &= \frac{1}{2} m\,(v(t_1))^2 - \frac{1}{2} m\,(v(t_0))^2 \end{aligned}$$

$=$ CHANGE IN <u>KINETIC</u> <u>ENERGY</u>

$$W = \tfrac{1}{2} m v^2_{FINAL} - \tfrac{1}{2} m v^2_{INITIAL}$$

$$W = (\text{KINETIC ENERGY})_{FINAL} - (\text{KINETIC ENERGY})_{INITIAL}$$

LET'S TAKE THIS ONE STEP FURTHER AND ASSUME THE FORCE $F(x)$ HAS A <u>POTENTIAL FUNCTION</u> $V(x)$, I.E., THAT

$$F(x) = -\frac{dV}{dx}$$

(THE MINUS SIGN IS CONVENTIONAL IN PHYSICS).

<u>EXAMPLE</u> : HOOKE'S LAW

$$F(x) = kx$$

$$= -\frac{dV}{dx}$$

WHERE $V(x) = -\tfrac{1}{2} k x^2$

THEN

$$W = \int_a^b F(x)\,dx = -\int_a^b \frac{dV}{dx}\,dx = -V(x)\Big|_a^b = V(a) - V(b)$$

= CHANGE IN <u>POTENTIAL ENERGY</u>

$$W = (\text{POTENTIAL ENERGY})_{INITIAL} - (\text{POTENTIAL ENERGY})_{FINAL}$$

EQUATING THESE TWO EXPRESSIONS FOR W GIVES

$$V(a) - V(b) = \frac{1}{2} m v_{FINAL}^2 - \frac{1}{2} m v_{INITIAL}^2$$

$$\frac{1}{2} m v_{INITIAL}^2 + V_{INITIAL} = \frac{1}{2} m v_{FINAL}^2 + V_{FINAL}$$

$$(\text{KINETIC ENERGY})_{INITIAL} + (\text{POTENTIAL ENERGY})_{INITIAL} =$$

$$(\text{KINETIC ENERGY})_{FINAL} + (\text{POTENTIAL ENERGY})_{FINAL}$$

OR

$$(\text{TOTAL ENERGY})_{INITIAL} = (\text{TOTAL ENERGY})_{FINAL}$$

THE TOTAL ENERGY IS UNCHANGED, I.E., CONSERVED.

THIS IS ONE INSTANCE OF A VERY POWERFUL PRINCIPLE IN PHYSICS CALLED

<u>CONSERVATION</u> <u>OF</u> <u>ENERGY</u>